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Secteur de recherche : Mathématiques et leurs interactions

Présentée par :
Dima Al Tabbaa

On the classification of some automorphisms of K3 surfaces

Directeur(s) de Thèse :
Alessandra Sarti

Soutenue le 07 décembre 2015 devant le jury

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Contents

Introduction.

Automorphisms of complex $K3$ surfaces have been widely studied in the last years, in particular also for the recent relation with the Bloch conjecture, see e.g. [28], [27]. In this thesis, we investigate (purely) non-symplectic automorphisms of order n, i.e. automorphisms that multiply the non-degenerate holomorphic two form by a primitive nth root of unity.
The study of non-symplectic automorphism of prime order was completed by Nikulin in [9] in the case of involutions, and more recently by Artebani, Sarti and Taki in several papers [3, 2, 16] for the other prime orders. The study of non-symplectic automorphisms of non-prime order turns out to be more complicated. Indeed, in this situation the "generic" case does not imply that the action of the automorphism is trivial on the Picard group [25, Section 11]. In the paper [14], Taki completely describes the case when the order of the automorphism is a prime power and the action is trivial on the Picard group. If we consider non-symplectic automorphisms of order 2^t, then by results of Nikulin we have $0 \leq t \leq 5$, and by a recent paper by Taki [15] there is only one $K3$ surface that admits a non-symplectic automorphism of order 32. Some further results in this direction are contained in a paper by Schütt [7] in the case of automorphisms of a 2-power order and in a paper by Artebani and Sarti [2], in the case of order 4. In this last paper, the hypothesis of trivial action on the Picard group is disregarded.

This thesis mainly deals with purely non-symplectic automorphisms σ of order eight and sixteen, which are quite unexplored, under the assumption that their fourth power σ^4 and eighth power σ^8 is the identity on the Picard lattice. By the Torelli type theorem, this holds for the generic element of the family of $K3$ surfaces carrying an order 8 and 16 non-symplectic automorphism with given action on its second cohomology group. The fixed locus $\text{Fix}(\sigma)$ of such an automorphism σ is the disjoint union of smooth curves and points.

In the first part of the thesis, we classify the non-symplectic automorphisms σ of order eight on a $K3$ surface when the fixed locus of its fourth power σ^4 contains a curve of positive genus. More precisely we show that the genus of the fixed curve by σ is at most one. After that, we study the case of the fixed locus of σ that contains at least a curve and all the curves fixed by its fourth power σ^4 are rational. Finally, we investigate the case when σ and its square σ^2 act trivially on the Néron-Severi group. We classify all the possibilities for the fixed locus of σ and σ^2 in these three cases. We obtain a complete classification for the non-symplectic automorphisms of order 8 on a $K3$ surface. More precisely, let X be a $K3$ surface, ω_X a generator of $H^{2,0}(X)$, σ an order 8 automorphism such that $\sigma^*\omega_X = \zeta_8\omega_X$, where ζ_8 denotes a primitive 8th root of unity. We denote by r, l, m and m_1 the rank of the eigenspaces of σ^* in $H^2(X, \mathbb{C})$ relative to the eigenvalues

$1, -1, i$ and ζ_8 respectively. We also denote by k the number of smooth rational curves fixed by σ, by N the number of isolated points in $\mathrm{Fix}(\sigma)$, and by $N^{'}$ the number of fixed points by σ contained in a curve $C \subseteq \mathrm{Fix}(\sigma^4)$ of genus $g \geq 1$. Finally, we denote by $2a$ the number of exchanged smooth rational curves by σ and fixed (that means pointwisely fixed) by σ^2. Thus we prove the following result.

Theorem 0.0.1. *Let σ be a purely non-symplectic order eight automorphism on a $K3$ surface X such that ${\sigma^*}^4$ acts identically on Pic(X). Then:*

$$n_{2,7} + n_{3,6} = 2 + 4\alpha, \;\; n_{4,5} + n_{2,7} - n_{3,6} = 2 + 2\alpha, \;\; N = 2 + r - l - 2\alpha.$$

Here we denote by $n_{i,j}$ the number of isolated points of type $P^{i,j}$ which are fixed by σ, and by $\alpha = \sum_{C \subset \mathrm{Fix}(\sigma)} 1 - g(C)$. Moreover, if Fix$(\sigma^4)$ contains a curve C of genus $g(C) = 1$, then the following possibilities hold.

- *If Fix(σ) contains the elliptic curve C, then*

$$(k, N, a, C^{'}) = (0, 2, 0, I_0) \;,\; (0, 4, 1, IV^*).$$

- *If Fix(σ^2) contains the elliptic curve C and there is no elliptic fixed curve by σ, then*

$$(k, N, a, N^{'}, C^{'}) = (0, 2, 0, 0, I_0),\; (0, 6, 0, 4, I_0),\; (0, 4, 0, 0, IV^*),\; (1, 10, 0, 4, IV^*).$$

- *If σ^i for $i = 1, 2$ does not fix an elliptic curve, then*

$$(k, N, a, N^{'}, C^{'}) = (0, 2, 0, 2, I_0),\; (1, 8, 0, 2, I_8),\; (0, 6, 0, 2, I_8),\; (0, 2, 1, 2, I_8),\; (0, 2, 0, 2, I_8),$$

$$(2, 14, 0, 2, I_{16}),\; (0, 6, 1, 2, I_{16}),\; (0, 2, 2, 2, I_{16}),\; (0, 2, 0, 2, I_{16}),\; (0, 4, 0, 0, IV^*),$$

where $C^{'}$ is another preserved fiber of the induced $\sigma-$invariant elliptic fibration that has the curve C as a fiber.

We prove Theorem 0.0.1 in several steps, mainly in Theorem 2.3.2, 2.3.3 and 2.3.4. We give examples showing the existence of all these cases except four of them in examples 2.8.1, 2.8.2, 2.8.3 and the Example 2.8.4 where we study a translation of order two acting on a generic fiber of an elliptic fibration.

If σ^* acts as the identity on Pic(X) (i.e. $l = m = 0$), then the fixed locus of σ contains points and at least a smooth rational curve, the rank of Pic(X) is either 6 or 14 and we have the following possibilities for (Pic$(X), N, k$):

$$(U \oplus D_4, 6, 1),\; (U(2) \oplus D_4, 6, 1) \text{ or } (U \oplus D_4 \oplus E_8, 12, 2).$$

We prove this result in Theorem 2.4.1 and Theorem B.0.9. More generally, if the order four automorphism σ^2 acts trivially on Pic(X) and $l \neq 0$, i.e. the action of σ on Pic(X) is not trivial, then the invariant of the fixed locus Fix(σ) are given in Table 2.4 and Table 2.3 for the case $g = 1$. We construct several examples corresponding to several cases with the assumption $m = 0$ in examples 2.3.4, 2.8.5, 2.8.6, 2.8.7, 2.8.8, 2.8.9 and 2.8.10.

Finally, we show the following theorem, where we use the same notation as before.

Theorem 0.0.2. *Let σ be a purely non-symplectic order eight automorphism on a $K3$ surface X such that ${\sigma^*}^4$ acts identically on Pic(X). Then the following hold.*

- *If Fix(σ) contains a curve then its genus is either* 0 *or* 1. *Moreover, let $g = g(C)$ be the genus of the curve $C \subset$ Fix(σ^4). Then*
  - *if $C \subset$ Fix(σ^2) then $g = 2, k = 0, N = 4$ and $r = 3, 13$.*
  - *if $C \not\subset$ Fix(σ^2) and $k > 0$, then $k = 1$ and $(g, N, r) = (2, 10, 13), \ (3, 6, 7), \ (3, 6, 8)$.*
- *If Fix(σ) contains a curve of genus $g = 0$ and all the curves fixed by σ^2 are rational, then we have only one possibility $(k, N, a, r) = (1, 10, 0, 13)$.*
- *If Fix(σ) is zero-dimensional, then it contains at most 6 points. If $m > 0$, then the possible invariants of Fix(σ) are given in Table 2.8 and we have $2 \leq r \leq 9, \ a = 0, 1 \ , \ g \leq 5$, where g is the highest genus of a curve fixed by σ^4.*

We prove this result in Theorem 2.5.1, 2.5.6, 2.6.1 and 2.7.1. In Example 2.8.9 we give two examples corresponding to the case $g = 0, k > 0$ and one case in Table 2.8 for $k = 0$.

In the second part of the thesis, we classify $K3$ surfaces with non-symplectic automorphism of order 16 in full generality.
Since Euler's totient function value of 16 must divide the rank of the transcendental lattice (see [8, Theorem 3.1]) the rank of the Picard group can only equal 6 or 14. More precisely, let X be a K3 surface, ω_X a generator of $H^{2,0}(X)$, σ an order 16 automorphism such that $\sigma^* \omega_X = \zeta_{16} \omega_X$, where ζ_{16} denotes a primitive 16th root of unity. We first show that if the fixed locus of σ contains a curve then its genus is zero (Proposition 3.2.3). We also show that the fixed locus of σ^4 always contains at least a curve of genus 0 or 1 (Proposition 3.1.10).
When $\operatorname{rk}\operatorname{Pic}(X) = 6$ and σ^8 acts trivially on $\operatorname{Pic}(X)$ (this is the generic case) we have the following number of isolated fixed points N and fixed rational curves k for σ (Theorem 3.3.2):

$$(\operatorname{Pic}(X), N, k) = (U \oplus D_4, 6, 1), \text{ or } (U(2) \oplus D_4, 4, 0).$$

In the first case, the action of σ is trivial on $\operatorname{Pic}(X)$ but not in the second case. If σ^8 acts trivially on $\operatorname{Pic}(X)$ (this is the generic case), we have $\operatorname{rk}\operatorname{Pic}(X) = 14$ and σ^4 fixes an elliptic curve C, then σ leaves C invariant (but C is not pointwise fixed by σ by Proposition 3.2.3) and the induced σ-invariant elliptic fibration has a reducible fiber of type IV^* (see [2, Theorem 3.1]). The number of isolated fixed points and fixed rational curves are as follows: $(N, k) = (8, 1)$ or $(6, 0)$. In the first case σ preserves each component of the fiber IV^* and in the second case it acts as a reflection on it. In any case the action of σ is nontrivial on $\operatorname{Pic}(X)$ (see Theorem 3.2.2). Finally if σ^8 acts trivially on $\operatorname{Pic}(X)$, $\operatorname{rk}\operatorname{Pic}(X) = 14$ and $\operatorname{Fix}(\sigma^4)$ contains at least a curve of genus zero we have three cases with $(\operatorname{Pic}(X), N, k)$ equal to:

$$(U \oplus D_4 \oplus E_8, 12, 1), \ (U(2) \oplus D_4 \oplus E_8, 4, 0) \text{ or } (U(2) \oplus D_4 \oplus E_8, 10, 1).$$

In these three cases the action of σ is not trivial on $\operatorname{Pic}(X)$, (Theorem 3.4.1). This in particular shows that there does not exist a K3 surface X with Picard number 14 with an automorphism of order 16 acting non-symplectically on it and trivially on $\operatorname{Pic}(X)$. This corrects a small mistake in the paper [14], where the author claims that such a K3 surface exists.

We construct the K3 surfaces in the Examples 3.5.1, 3.5.2, 3.5.3 (some of the examples are described in [14] and [26]). For the proofs of the Theorems 3.2.2, 3.3.2, 3.4.1, we use

Lefschetz formulas and results on non-symplectic involutions and on non-symplectic order four automorphisms, which are contained in [2], [14]. We also use some results (Lefschetz formulas, local action at the fixed points) on non-symplectic automorphisms of order eight.

The structure of this thesis is as follows. In **Chapter 1**, we introduce basic facts about lattices, $K3$ surfaces and non-symplectic automorphism of finite order on $K3$ surfaces, and about elliptic fibrations over $K3$ surfaces.

In **Chapter 2**, we classify the non-symplectic automorphisms σ of order eight on a $K3$ surface X. In Section 2.1, we give a general description of the fixed locus of σ. By means of Lefschetz's formulas, we provide relations between the invariants N, k, g and the ranks of the eigenspaces of σ^* on the lattice $H^2(X, \mathbb{C})$. In Section 2.2, we investigate elliptic fibrations $\pi : X \longrightarrow \mathbb{P}^1$ such that σ^4 fixes a curve C of genus $g > 1$ and $\mathrm{Pic}(X) \cong S(\sigma^4) = U \oplus L$.
In Section 2.3, we suppose that σ^4 fixes an elliptic curve C and we describe the singular fibers of the elliptic fibration with fiber C and the corresponding structure of the fixed locus of σ. Here we distinguish the following three cases.

- The elliptic curve C is fixed by σ (Theorem 2.3.2).
- The elliptic curve C is fixed by σ^2 but is not contained in $\mathrm{Fix}(\sigma)$ (Theorem 2.3.3).
- The elliptic curve C is only fixed by σ^4 (Theorem 2.3.4).

In Section 2.4, we assume that σ^2 acts as the identity on $S(\sigma^4) \cong \mathrm{Pic(X)}$, i.e. $m = 0$. In Section 2.5 and 2.6, we suppose that σ fixes at least one rational curve and we classify the cases when the fixed locus of σ^4 contains a curve of genus $g > 1$ or it contains only rational curves. In Section 2.7, we consider the case when σ fixes only isolated points. In Section 2.8, we give several examples corresponding to several cases in the classification of non-symplectic automorphisms of order eight. We construct all these examples by elliptic fibrations over $K3$ surfaces.

In **Chapter** 3, we classify $K3$ surfaces with non-symplectic automorphisms of order 16. In Section 3.1, we give a general description of the fixed locus of σ and we recall some useful facts. We show more precisely that $\mathrm{rk}\ \mathrm{Pic}(X) = \mathrm{rk}\ S(\sigma^8)$ is either 6 or 14. In Section 3.2, we suppose that σ^8 fixes an elliptic curve C. In Section 3.3, we study the case when $\mathrm{Pic}(X) = S(\sigma^8)$ has rank 6. In Section 3.4, we assume that the rank of the Néron-Severi group is 14. Finally, in Section 3.5 we give an example for each case in the classification of the non-symplectic automorphisms of order 16.

Finally, in Appendix A we classify all quartic surfaces that are affinely invariant for the action of some automorphism of order 8 of $\mathbb{P}^3$ acting non-symplecticaly on the quartic. In Appendix B, we assume that $l = 0$, so that $r_{\sigma^2} = r$ (i.e. σ^* acts as the identity on $S(\sigma^2)$). More precisely, we study the case when σ^* is the identity on the Picard lattice and we give an independent proof (not based on the classification of order four automorphism [2]) of [14, Proposition 5.5]. In Appendix C, we give the tables for the complete classification of the non-symplectic automorphisms of order 8 on a $K3$ surface. These show the invariants of non-symplectic automorphisms of order 8 in all the possible cases. Moreover, they also show the cases when we have an example or when there is not an example (and we give the number of examples if we have more than one).

Chapter 1

Preliminaries

1.1 Lattices.

A **lattice** L is a free $\mathbb{Z}$-module of finite rank with a $\mathbb{Z}$-valued symmetric bilinear form:

$$b : L \times L \longrightarrow \mathbb{Z}.$$

An isomorphism of $\mathbb{Z}-$modules preserving the bilinear form is called **isometry**. The group of the isometries of L is indicated with $O(L)$.
The lattice L is said to be **even** if the quadratic form associated to b takes only even values. Otherwise, i.e. if the quadratic form associated to b takes only odd values, it is called **odd**.

The **discriminant** $d(L)$ of L is the determinant of a matrix associated to b, and L is said to be **unimodular** if $d(L) = \pm 1$. If L is **non-degenerate**, i.e. $d(L) \neq 0$, then the **signature** of L is a pair (s_+, s_-) of integers, where $s_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form on $L \otimes \mathbb{R}$; L is called **positive-definite** (**negative-definite**) if the quadratic form associated to b takes only positive (negative) values. If the signature of the lattice L is $(1, \mathrm{rank(L)} - 1)$, then L is called **hyperbolic**.

We define the **dual of the lattice** L to be:

$$L^\vee = Hom_{\mathbb{Z}}(L, \mathbb{Z}) \cong \{v \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid b(v, x) \in \mathbb{Z} \text{ for all } x \in L\}.$$

There is a natural embedding, the **correlation morphism**, of L in $L^\vee$ via $c \mapsto b(c, -)$.
The **discriminant group** of a lattice L is the group $A_L = L^\vee / L$. Let A be a finite abelian group. The **length** of A, $l(A)$, is the minimum number of generator of A.
Let $b^{'} : A \times A \longrightarrow \mathbb{Q}/\mathbb{Z}$ be a symmetric even bilinear form. It induces a quadratic form q on the group A such that:

- $q : A \longrightarrow \mathbb{Q}/2\mathbb{Z}$,
- $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$ and $a \in A$.
- $q(a + a^{'}) - q(a) - q(a^{'}) \equiv 2b(a, a^{'}) \mod 2.$

Let L be a non-degenerate even lattice with bilinear form b. The $\mathbb{Q}$-linear extension of b to $L^\vee$ is a symmetric bilinear form

$$L^\vee \times L^\vee \longrightarrow \mathbb{Q}.$$

This form induces a symmetric bilinear form on the discriminant group of L, namely :

$$b_L : L^\vee/L \times L^\vee/L \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Let q_L be the quadratic form associated to b_L :

$$q_L : L^\vee/L \longrightarrow \mathbb{Q}/2\mathbb{Z}.$$

We call q_L the **discriminant form** of the lattice L.

We denote by $L(m)$ the lattice which coincides with L as $\mathbb{Z}$-module and with bilinear form multiplied by m with respect to the bilinear form of L. By $L^{\oplus k}$ we denote the lattice with bilinear form which is the orthogonal sum of k copies of the bilinear form b of L.
Let (L, b) and (M, b) be two lattices such that $M \subset L$. The lattice $M^{\perp L}$ is the sublattice of L given by

$$M^{\perp L} := \{l \in L \mid b(l, m) = 0 \text{ for each } m \in M\}.$$

Proposition 1.1.1. (see [22, Ch I, Lemma 2.1])
Let L be a non degenerate lattice, and let $\phi : L \longrightarrow L^\vee$ be the correlation morphism of the lattice L. Then we have the following:

- *The index of $\phi(L)$ in $L^\vee$ is $|d(L)|$.*
- *If M is a sublattice of the lattice L with* rank L = rank M, *then the square of the index of M in L is:*
$$[L : M]^2 = \left|\frac{d(M)}{d(L)}\right|.$$

Proposition 1.1.2. (see [10, Corollary 1.13.3]*)*
Let L be an even lattice with signature (s_+, s_-) and discriminant form q_L. If $s_+ > 0, s_- > 0$ and $l(A_L) \leq$ rank (L) $- 2$, then L is the only lattice with these invariants up to isometry.

Examples :

1) The lattice U is the unique rank two unimodular lattice of signature $(1, 1)$, whose Gram matrix is:
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2) The lattices A_n, D_k, E_i for $n \geq 1, k \geq 4, i = 6, 7, 8$ are the even, negative lattices associated with the Dynikin diagrams of the corresponding types.
The following Table 1.1 shows the form of Dynkin diagrams, the rank and the determinant of each lattice.

The lattices A_n, D_k, E_i for $n \geq 1, k \geq 4, i = 6, 7, 8$ are the even, negative definite lattices associated with the Dynkin diagrams of the corresponding types.
The following Table 1.1 shows the form of Dynkin diagrams, the rank and the determinant of each lattice.

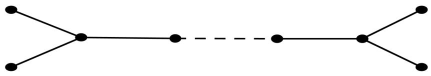
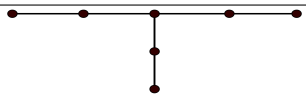
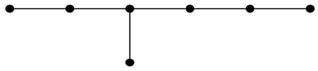
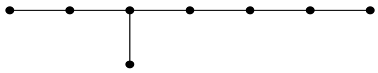
name	rank	det	associated Dynkin diagram
A_n	n	$n+1$	
D_k	k	4	
E_6	6	3	
E_7	7	2	
E_8	8	1	

Table 1.1: Dynkin diagram

The Gram matrices associated with these lattices are the following:

$$A_n = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & -2 \end{pmatrix}, \quad D_k = \begin{pmatrix} -2 & 0 & 1 & & & & \\ 0 & -2 & 1 & & & & \\ 1 & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & 1 \\ & & & 1 & -2 & \\ & & 1 & & & -2 \end{pmatrix}, \quad E_7 = \begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & 1 \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & \\ & & 1 & & & & -2 \end{pmatrix}$$

$$E_8 = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & 1 \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & 1 & & & & & -2 \end{pmatrix}$$

Definition 1.1.3. *Let p be a prime number. A lattice L is called p-elementary if $A_L \simeq (\mathbb{Z}/p\mathbb{Z})^a$; $a \in \mathbb{Z}_{\geq 0}$.*

Remark 1.1.4. *If L is a p-elementary lattice primitively embedded in a unimodular lattice M and $L^{\perp M}$ is its orthogonal complement in M, then it is known that $L^{\perp M}$ is also $p-$elementary and $p^a = |det(L)| = \left|det(L^{\perp M})\right|$.*

The following result classifies even, indefinite, p-elementary lattices (see [19]. section 1).

Theorem 1.1.5. *An even, indefinite, p-elementary lattice of rank r for $p \neq 2$ and $r \geq 2$ is uniquely determined by the integer a.*

For $p \neq 2$ a hyperbolic p-elementary lattice with invariants a, r exists if and only if the following conditions are satisfied: $a \leq r$, $r \equiv 0 \mod 4$ and

$$\begin{cases} for & a \equiv 0 \mod 2 \quad r \equiv 2 \mod 4 \\ for & a \equiv 1 \mod 2 \quad p \equiv (-1)^{r/2-1} \mod 4. \end{cases}$$

Moreover $r > a > 0$, if $r \not\equiv 2 \mod 8$.

Example 1.1.6. (See [4, Section 1])

- *If $p \equiv 1 \mod 4$, then the lattice defined by the following matrix:*

$$H_p = \begin{pmatrix} -(p+1)/2 & 1 \\ 1 & -2 \end{pmatrix}.$$

  is negative definite, p-elementary with $a = 1$.

- *If $p \equiv 1 \mod 4$ then the lattice given by:*

$$H_p = \begin{pmatrix} (p-1)/2 & 1 \\ 1 & -2 \end{pmatrix}.$$

  is hyperbolic, $p-$elementary with $a = 1$.

- *We return to the lattices in Table 1.1. For a prime number p the A_{p-1} lattice is a $p-$elementary lattice with length $a = 1$, while the A_1, D_{2k}, E_7 and E_8 lattices are 2-elementary lattices with a equal to 1,2,1 and 0 respectively. Observe for example that the lattice $A_1 \oplus D_4^{\oplus 2} \oplus E_8$ is a 2-elementary lattice with length $a = 1+2+2+0 = 5$. Finally, the A_2 and E_6 lattices in Table 1.1 are 3-elementary lattices.*

Remark 1.1.7. *An even, indefinite, 2-elementary lattice is determined by the rank r and the length a and by a third invariant $\delta \in \{0, 1\}$, see [9].*

1.2 K3 surfaces and non-symplectic automorphisms.

Here we introduce basic facts about $K3$ surfaces, the $K3-$lattice, the Néron-Severi group and the transcendental lattice. Then we define the non-symplectic automorphisms of finite order n on $K3$ surfaces and we give their main properties.

1.2.1 K3 surfaces.

Definition 1.2.1. *A **K3 surface** is a compact complex surface X with $q = h^{1,0}(X) = 0$ and $K_X \sim 0$, where K_X is the canonical divisor on X and $\sim$ is the linear equivalence.*

Proposition 1.2.2. (see [22, Ch VIII])
All the $K3$ surfaces are diffeomeorphic. Moreover, we have that:

- *They are all simply connected and their Betti numbers are $b_0 = 1, b_1 = 0, b_2 = 22$.*
- *All the $K3$ surfaces are Kähler surfaces.*
- *The Hodge numbers of a $K3$ surface are $h^{2,0} = h^{0,2} = 1, h^{1,1} = 20, h^{1,0} = h^{0,1} = 0$ and the Euler characteristic is 24.*

Proposition 1.2.3. *Let X be a $K3$ surface. Then the Picard and the Néron-Severi group of X coincide $NS(X) \simeq \mathrm{Pic(X)}$.*

Proof. For the complex surface X the exponential sequence induces the following exact sequence:

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}).$$

Since X is a $K3$ surface $H^1(X, \mathcal{O}_X) = 0$ and so the morphism c_1 is an injective map. Thus $c_1(\mathrm{Pic}(X)) \simeq \mathrm{Pic}(X)$. This means that the **Néron-Severi group** $NS(X)$, defined for every surface S as $\mathrm{Pic}(S)/\ker(c_1)$, coincides with the Picard group $\mathrm{Pic}(X)$ if the surface is of type $K3$. In particular, the linear and the numerical equivalence coincide. □

Since $H^2(X, \mathbb{Z})$ is torsion free it is a lattice with the pairing induced by the cup product, and by the previous exact sequence $NS(X)$ is a sublattice of it. The pairing on $NS(X)$ coincides with the intersection form on $\mathrm{Pic}(X)$.

The orthogonal to the lattice $NS(X)$ in $H^2(X, \mathbb{Z})$ is the **transcendental lattice**. We will denote it by $T_X = NS(X)^{\perp_{H^2(X,\mathbb{Z})}}$.

More precisely, the lattice $H^2(X, \mathbb{Z})$ with the cup product is an even unimodular lattice of signature $(3, 19)$. Up to isometrie there exists only one lattice with these properties that is $U^{\oplus 3} \oplus E_8^{\oplus 2}(-1)$. So the second cohomology group $H^2(X, \mathbb{Z})$ of any $K3$ surface is isometric to the $K3$ **lattice**:

$$\Lambda_{K3} = U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1),$$

where $E_8(-1)$ denotes the lattice whose entries of the Gram matrix are the opposite of those given for the E_8-lattice.

Remark 1.2.4.
- *Since $h^{2,0}(X) = 1$, the lattice $H^{2,0}(X)$ is generated by an element ω_X called a period of the $K3$ surface X. It satisfies the conditions $\langle \omega_X, \omega_X \rangle = 0$ and $\langle \omega_X, \overline{\omega}_X \rangle > 0$.*
- *By the Hodge index theorem, the signature of the Néron-Severi group of an algebraic $K3$ surface is $(1, \rho - 1)$, where ρ is the Picard number, i.e. the rank of $NS(X)$. Then the transcendental lattice of an algebraic $K3$ surface has signature $(2, 20 - \rho)$. We can identify the Néron-Severi group of a $K3$ surface X with the set:*

$$NS(X) \simeq Pic(X) = \{x \in H^2(X, \mathbb{Z});\ \langle x, \omega_X \rangle = 0\}.$$

For each $\alpha \in \Lambda_{K3}$ we denote with $[\alpha] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$ the corresponding line. The **period domain** of $K3$ surface is the set

$$\Omega = \{[\omega] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \text{ such that } \langle \omega_X, \omega_X \rangle = 0 \text{ and } \langle \omega_X, \overline{\omega}_X \rangle > 0\}.$$

We recall now the formula for the genus of irreducible smooth curves on smooth surfaces.

Lemma 1.2.5. *Let C be an irreducible smooth curve on the surface X. Then the genus of C is given by:*

$$g(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_X).$$

If X is a $K3$ surface, then we have seen that the canonical divisor K_X is trivial, so that we get:

$$g(C) = 1 + \frac{1}{2}C^2.$$

Consider now two curves $C, C^{'}$ and let $f : C \longrightarrow C^{'}$ be a finite, separable morphism of degree n. The **Riemann-Hurwitz formula** describes the relationship of the Euler characteristics of two curves and gives a relation between their generas.

Lemma 1.2.6. (See [23, Corollary 2.4])
With the previous hypothesis we have that:

$$2 - 2g(C) = n(2 - 2g(C^{'})) - \sum_{p \in C}(e_p - 1), \tag{1.2.1}$$

where e_p is the ramification index at a ramification point $p \in C$.

An isometry of the lattice $H^2(X, \mathbb{Z})$ is a **Hodge isometry** if its $\mathbb{C}$-linear extension preserves the Hodge decomposition of $H^2(X, \mathbb{C})$. In particular, if X and Y are $K3$ surfaces a Hodge isometry i between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ is an isometry between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ such that the $\mathbb{C}$-linear extension of i preserves the Hodge decomposition.

The positive cone of X, $V(X)^+$, is the connected component of $V(X) = \{x \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$ such that $(x, x) > 0\}$ containing a Kähler class (this implies that it contains all the Kähler classes). An isometry between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ is called **effective** if it preserves the positive cone and induces a bijection between the respective sets of effective divisors.

Proposition 1.2.7. *Let $f : X \longrightarrow Y$ be an isomorphism between $K3$ surfaces. Then $f^* : H^2(Y, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z})$ is an effective Hodge isometry. In particular, an automorphism of a $K3$ surface X induces an effective Hodge isometry on $H^2(X, \mathbb{Z})$.*

Theorem 1.2.8. *(Torelli theorem for $K3$ surfaces) Let X and Y be two $K3$ surfaces. Suppose that there exists an effective Hodge isometry $\varphi : H^2(Y,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z})$. Then there exists a unique isomorphism $f : X \longrightarrow Y$ such that $\varphi = f^*$.*

1.2.2 Non-symplectic automorphisms.

Let X be a $K3$ surface. Recall by Remark 1.2.4 that the vector space $H^{2,0}(X)$ is generated by a nowhere vanishing holomorphic two-form ω_X, i.e $H^{2,0}(X) = \mathbb{C}\omega_X$. Let $\sigma \in \mathrm{Aut}(X)$ be an automorphism on a $K3$ surface X. The action of σ on the $K3$-lattice $\sigma^* : H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z})$ is an effective Hodge isometry (see Proposition 1.2.7). This means that the Hodge structure of the surface X is preserved by σ^* and so we have $\sigma^*(\omega_X) = \lambda_\sigma\omega_X$ for $\lambda_\sigma \in \mathbb{C}^*$. The *Torelli* theorem for $K3$ surfaces (see Theorem 1.2.8) shows that the action of the automorphism σ is uniquely determined by σ^*.

Definition 1.2.9. *A* ***non-symplectic automorphism*** *of finite order n on a $K3$ surface X is an automorphism $\sigma \in Aut(X)$ that satisfies $\sigma^*(\omega_X) = \lambda_\sigma\omega_X$ with $\lambda_\sigma \neq 1$ (if $n = 2$ we call σ a non-symplectic* ***involution****), otherwise we call σ* ***symplectic****. More precisely, we assume that $\sigma^*(\omega_X) = \zeta_n\omega_X$, where $\zeta_n = e^{\frac{2\pi i}{n}}$ is a primitive nth root of the unity. We call σ a* ***purely non-symplectic*** *automorphism.*

Remark 1.2.10. *Since in this thesis we only work with purely non-symplectic automorphisms, we call them non-symplectic for simplicity.*

This thesis is devoted to the study of non-symplectic automorphisms of <u>even order</u> on $K3$ surfaces and especially to non-symplectic automorphisms of order eight and sixteen (i.e. $n = 8, 16$ in Definition 1.2.9).
We denote by $\mathrm{Fix}(\sigma)$ the **fixed locus** of the automorphism σ such that:

$$\mathrm{Fix}(\sigma) = \{x \in X \mid \sigma(x) = x\}.$$

We can find easily that $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\sigma^i)$ for $i = 2, 3, \dots, n-1$.
The **invariant lattice** of σ is given by:

$$S(\sigma) = \{x \in H^2(X,\mathbb{Z}) \mid \sigma^*(x) = x\}.$$

Proposition 1.2.11. *Let X be a $K3$ surface with a purely non-symplectic automorphism σ of finite order n. Then $\mathrm{rk}\ \mathrm{S}(\sigma) > 0$ and $S(\sigma) \subseteq \mathrm{Pic(X)}$.*

Proof. First, observe that rk $S(\sigma) > 0$ since there is always an ample invariant class on X (see [8, Theorem 3.1]).
On the other hand, let $v \in S(\sigma)$ then $\sigma(v) = v$, now the intersection product can be extended to $H^2(X,\mathbb{Z}) \otimes \mathbb{C}$ so we have :

$$\langle v, \omega_X\rangle \;=\; \langle \sigma^*(v), \sigma^*(\omega_X)\rangle \;=\; \langle v, \zeta_n\omega_X\rangle,$$

so that $\langle v, \omega_X\rangle = 0$. By the description of $\mathrm{Pic}(X) = \{\omega_X^\perp \cap H^2(X,\mathbb{Z})\}$ (see Remark 1.2.4) we get that $v \in \mathrm{Pic}(X)$ so that $S(\sigma) \subseteq \mathrm{Pic}(X)$. □

Remark 1.2.12. *Let ν be a non-symplectic involution on X. In the generic case we can assume that $Pic(X) = S(\nu)$, i.e. the action of the involution ν is trivial on $Pic(X)$.*

On the other hand, let

$$T(\sigma) = S(\sigma)^{\perp} \cap H^2(X, \mathbb{Z}).$$

Since $S(\sigma) \subseteq \mathrm{Pic}(X)$ (see Proposition 1.2.11), the transcendental lattice satisfies that $T_X \subseteq T(\sigma)$. Recall that the action of σ on T_X and $T(\sigma)$ is by primitive roots of the unity, see [8, Theorem 3.1 (c)], moreover the following proposition holds:

Proposition 1.2.13. *Let σ be a non-symplectic automorphism of even order n on a $K3$ surface X. Then all of the eigenvalues of σ^* on $H^2(X, \mathbb{Z}) \otimes \mathbb{C}$ (which are the primitive nth roots of the unity) have the same multiplicity q.*

Proof. Consider the restriction of σ^* to T_X. By [8, Theorem 3.1 (c)] the eigenvalues for the action of σ^* on $T_X \otimes \mathbb{C}$ are primitive nth roots of unity. Since σ^* is an isometry of T_X, the characteristic polynomial $P(X)$ of σ^* is in $\mathbb{Q}[X]$. The minimal polynomial of a primitive nth root of unity is $X^{n/2} + 1$, hence $P(X) = (X^{n/2} + 1).\overline{P}(X)$ where $\overline{P}(X) \in \mathbb{Q}[X]$. The roots of $\overline{P}(X)$ are again primitive roots of unity of order n by [8, Theorem 3.1 (c)]. Hence one can conclude that $P(X) = (X^{n/2} + 1)^q$, this proves the statement. □

Remark 1.2.14. *Observe that with a little modification in the proof of the previous proposition the statement also holds for non-symplectic automorphisms with odd order n.*

The **moduli space** of $K3$ surfaces carrying a non-symplectic automorphism of even order n ($n \neq 2$) with a given action on the $K3$ lattice is known to be a complex ball quotient of dimension $q - 1$ where q is rank of the eigenspace of σ^* in $H^2(X, \mathbb{C})$ relative to the eigenvalues $\zeta_n = e^{\frac{2\pi i}{n}}$, see [25, §11]. The complex ball is given by:

$$B = \{[w] \in \mathbb{P}(V) : (w, \overline{w}) > 0\},$$

where V is the ζ_n-eigenspace of σ^* in $T(\sigma^{n/2}) \otimes \mathbb{C}$. This implies that the Picard group of a $K3$ surface corresponding to the generic point of such space equals $S(\sigma^{n/2})$ (see [25, Theorem 11.2]). This shows more precisely the case in Remark 1.2.12.

The action of the non-symplectic automorphism σ of finite order n on a neighborhood of a fixed point $x \in X$ can be locally linearized and diagonalized and so is given by a matrix of the form:

$$A_{i,j} = \begin{pmatrix} \zeta_n^i & 0 \\ 0 & \zeta_n^j \end{pmatrix}$$

for $i = 1, \dots, n$ and $i + j \equiv 1 \mod n$.

We distinguish two cases: if $(i, j) = (1, 0)$ (or $(i, j) = (0, 1)$) the matrix $A_{1,0}$ is

$$\begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix}$$

and then the point x belongs to a smooth fixed curve for σ (corresponding to the eigenvalue 1 in the matrix). Otherwise, if $i \neq 0, n$ (or $j \neq 0, n$) the point x is an isolated fixed point by σ and we say that x is a **fixed point of type** $P^{i,j}$. We recall now the following useful lemma which explain how we know the type of the fixed points in a tree of smooth rational curves. This result, which generalizes the Lemma 8.1 in [13], is proved in [2].

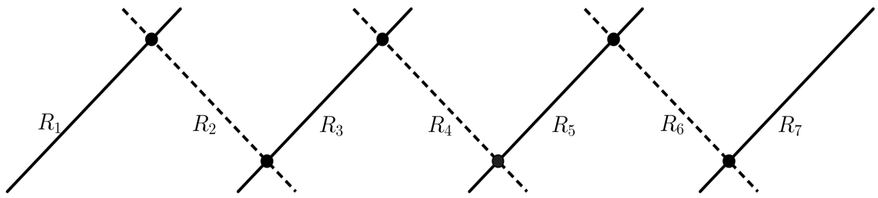


Figure 1.1: the action of an involution on a tree of smooth rational curves.

Lemma 1.2.15. *Let $T = \sum_i R_i$ be a tree of smooth rational curves on a $K3$ surface X such that each R_i is invariant under the action of a purely non-symplectic automorphism σ of order n. Then, the points of intersection of the rational curves R_i are fixed by σ and the action at one fixed point determines the action on the whole tree.*

Remark 1.2.16. *The local action at the intersection points of the curves R_i appear in the following order:*

$$\cdots\begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_n^{n-1} & 0 \\ 0 & \zeta_n^2 \end{pmatrix}, \begin{pmatrix} \zeta_n^{n-2} & 0 \\ 0 & \zeta_n^3 \end{pmatrix}, \ldots, \begin{pmatrix} \zeta_n^2 & 0 \\ 0 & \zeta_n^{n-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta_n \end{pmatrix}\cdots$$

Practically, we see by Lemma 1.2.15 and Remark 1.2.16 that: for $n = 2$ the fixed locus of a non-symplectic involution ν does not contain isolated point (since the local action of ν at a fixed point is of type $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$). So that for a tree of smooth rational curves, each curve is preserved by ν, we have the following. A curve between two invariant (not pointwise fixed) curves is fixed by ν pointwisely, while the curve which intersects two pointwise fixed curves is just invariant by ν with two fixed points on it (the intersection points with the two fixed curves).

Figure 1.1 shows the situation for a tree of seven smooth rational curves. We denote the invariant curve (not pointwise fixed) with a dotted line.

Non-symplectic involutions.

As we have seen previously the fixed locus for any non-symplectic automorphism σ of even order $n \geq 2$ on a $K3$ surface X, is a subset of the fixed locus of the involution $\nu := \sigma^{n/2}$. So that it is very useful to briefly recall the classification theorem for non-symplectic involution on $K3$ surfaces (see [4, §2]) which was given by Nikulin in [9, §4] and [11, §4].

The Fixed locus $\mathrm{Fix}(\nu)$ is the disjoint union of smooth curves and there are no isolated fixed points. The lattice $S(\nu)$ is 2-elementary (i.e. its discriminant group $A_{S(\nu)} = S(\nu)^\vee/S(\nu) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$) thus, according to Theorem 1.1.5 and Remark 1.1.7, its isometry class is determined by the invariants $r = \mathrm{rk}\ S(\nu) = \mathrm{rk}\ \mathrm{Pic}(X)$, the length a and δ (where we introduce the invariant δ of $S(\nu)$ by putting $\delta = 0$ if $x^2 \in \mathbb{Z}$ for any $x \in A_{S(\nu)}$ and $\delta = 1$ otherwise).

Theorem 1.2.17. [9, Theorem 4.2.2]
The fixed locus of a non-symplectic involution on a $K3$ surface is

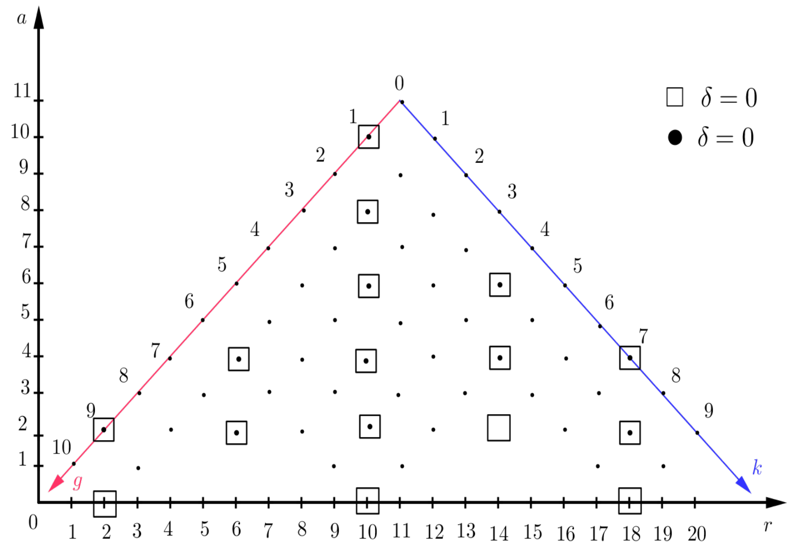


Figure 1.2: Non-symplectic involutions

- *empty if* $r = 10, a = 10$ *and* $\delta = 0$,
- *the disjoint union of two elliptic curves if* $r = 10, a = 8$ *and* $\delta = 0$,
- *the disjoint union of a curve of genus* g *and* k *rational curves otherwise, where*

$$g = (22 - r - a)/2, \quad k = (r - a)/2.$$

Figure 1.2 shows all the values of the triple (r, a, δ) which are achieved and the corresponding invariants (g, k) of the fixed locus.

1.3 Elliptic fibrations.

Definition 1.3.1. *Let X be a complex surface. A* ***genus one curve fibration*** *is a holomorphic map $f : X \longrightarrow B$ to a smooth curve B such that the generic fiber is a smooth connected curve of genus one. An* ***elliptic fibration*** *is a genus one curve fibration admitting a section $s : B \longrightarrow X$. The surface X is called an* ***elliptic surface****. We call F_v the fiber $f^{-1}(v)$ over a point $v \in B$.*
The ***Mordell Weil group*** *is the group of the sections of the elliptic fibration. The Mordell Weil group is indicated with $MW(X)$.*

The **zero section** of an elliptic fibration is a chosen section $s : B \longrightarrow X$ and we identify the map s with the curve $s(B)$ on X. The point of intersection between the zero section and a fiber is the zero of the group law on the fiber.

For $K3$ surfaces we have that $B = \mathbb{P}^1$ (see [17]) and the fibration admits a Weierstrass form:

$$y^2 = x^3 + \alpha(t)x + \beta(t), \tag{1.3.1}$$

where $\alpha(t)$ and $\beta(t)$ are two polynomials with complex coefficients such that deg $\alpha(t) = 8$ and deg $\beta(t) = 12$.
The **discriminant** of the fibration is a degree 24 polynomial:

$$\Delta(t) = 4\alpha(t)^3 + 27\beta(t)^2. \tag{1.3.2}$$

The equation 1.3.1 is associated with an elliptic fibration if and only if $\Delta(t)$ does not vanish identically.

Each root of $\Delta(t)$ corresponds to a point p of the base $\mathbb{P}^1$ such that F_p is a singular fiber of the fibration. There are at most finitely many singular fibers. Let δ be the order of vanishing of Δ in the point corresponding to the singular fiber, by *Kodaira classification* the possible singular fibers are:

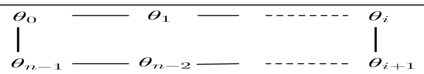
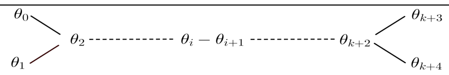
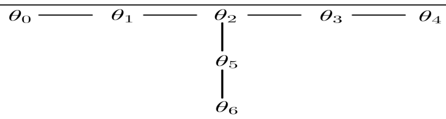
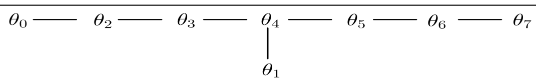
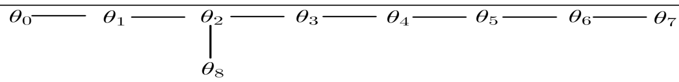
name	*Dynkin diagrams*	description	δ
II		a cuspidal rational curve	2
I_1		nodal rational curve	1
I_2	A_1	two rational curves meeting transversally at two points	2
$I_n, n \geq 3$	$\tilde{A}_n$		n
$I_n^*, n > 0$	$\tilde{D}_{k+4}$		$n+6$
III		two rational curves meeting in a point of order 2	3
IV		three rational curves all meeting at one point	4
IV^*	$\tilde{E}_6$		8
III^*	$\tilde{E}_7$		9
II^*	$\tilde{E}_8$		10

Table 1.2: Kodaira classification

The Euler characteristic of the singular fibers are

$$\begin{array}{llll} e(I_n) = n, & e(II) = 2, & e(III) = 3, & e(IV) = 3, \\ e(I_n^*) = n+6, & e(II^*) = 10, & e(III^*) = 9, & e(IV^*) = 8, \end{array}$$

where Θ_0 is the component of a fiber meeting the zero section. The first column in the Table 1.2 contains the name of the reducible fiber according to Kodaira classification, the second the Dynkin diagram associated to the fiber, the last column contains the order of vanishing of Δ in the point corresponding to the singular fiber.

Remark 1.3.2. *Let $F_v^{\#}$ be the subset of the fiber F_v obtained by deleting the singular points of the fiber, and let $F_{v,0}^{\#}$ be the subset of $F_v^{\#}$ obtained by deleting the component meeting the zero section. Then $F_v^{\#}$ is an Abelian group, with the operation induced by the* *operation of the Mordell Weil group. For each type of fiber we describe here the groups* $F_{v,0}^{\#}$ *and* $F_v^{\#}/F_{v,0}^{\#}$ *and so the group* $F_v^{\#}$*.*

Fiber	$F_{v,0}^{\#}$	$F_v^{\#}/F_{v,0}^{\#}$
I_0	*elliptic*	0
$I_n, n \geq 1$	$\mathbb{C}^*$	$\mathbb{Z}/n\mathbb{Z}$
I_n^*	$\mathbb{C}$	$\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ iseven} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n \text{ isodd} \end{cases}$
II, II^*	$\mathbb{C}$	0
III, III^*	$\mathbb{C}$	$\mathbb{Z}/2\mathbb{Z}$
IV, IV^*	$\mathbb{C}$	$\mathbb{Z}/3\mathbb{Z}$

In particular this gives strict conditions on the automorphism group of an elliptic fibration with a reducible fiber, because an automorphism of a reducible fiber has to be compatible with the group structure of the fiber.

A **simple component** of a fiber is a component with multiplicity one. In the following table we describe the singular fibers of an elliptice fibration with the multiplicities of the vertices of the extended Dynkin diagrams, and we indicate the simple components.

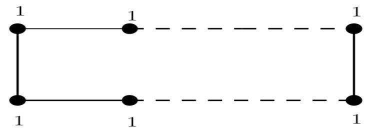
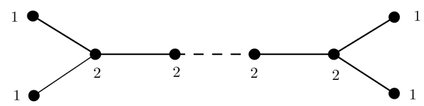
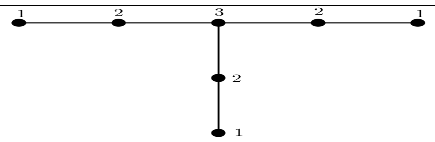
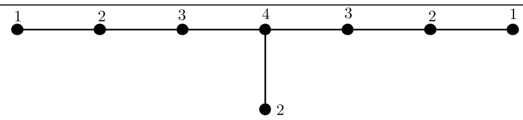
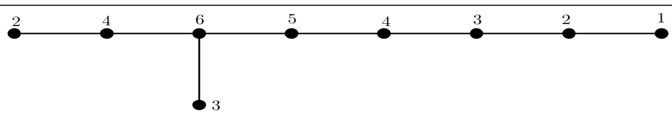
name	simple components	associated Dynkin diagram
$\tilde{A}_n$	$\Theta_i, \quad i = 0, \ldots, n-1$	
$\tilde{D}_{k+4}$	$\Theta_i, \quad i = 0, 1, k+3, k+4$	
$\tilde{E}_6$	$\Theta_i, \quad i = 0, 4, 6$	
$\tilde{E}_7$	$\Theta_i, \quad i = 0, 7$	
$\tilde{E}_8$	Θ_7	

Table 1.3: Dynkin diagrams with the multiplicities of the components

The Néron-Severi group of a surface admitting an elliptic fibration contains the class of a fiber F (all the fibers are algebraically equivalent) and the class of the zero section s. Since the fibers are all algebraically equivalent $\underline{F \cdot F = 0}$. The zero section intersects any fiber in one point, so that $\underline{F \cdot s = 1}$.
The sections of an elliptic fibration on $K3$ surfaces are rational curves and this implies that

their self-intersection is -2. Moreover, if X is a $K3$ surface admits an elliptic fibration, then there is an embedding of $\bar{U}$ in $NS(X)$, where $\bar{U}$ is the two dimensional lattice

$$\bar{U} = \left\{ \mathbb{Z}^2, \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \right\}.$$

Observe that the lattice $\bar{U}$ is isometric to the hyperbolic plane U.

We recall now the **Shioda-Tate formula** (see [20, Section 7]): Let Red be the set $Red = \{v \in \mathbb{P}^1 | F_v \text{ is reducible}\}$. Let ρ be the Picard number of the surface X and m_v is the number of irreducible components of the fiber F_v. Then

$$\rho(X) = \text{rk } NS(X) = \text{rk } MW(X) + 2 + \sum_{v \in Red} (m_v - 1).$$

The Néron-Severi lattice of X is generated over $\mathbb{Q}$ by F, by the classes of irreducible components of the reducible fiber, which do not intersect the zero section and by the sections.

Definition 1.3.3. *The **trivial lattice** Tr_X of an elliptic fibration on a surface is the lattice generated by the class of the fiber, the class of the zero section and the irreducible classes of the reducible fibers which do not intersect the zero section.*

Remark 1.3.4. *The lattice Tr_X admits $\bar{U}$ as sublattice and its rank is*

$$rk\ T_X = 2 + \sum_{v \in Red} (m_v - 1).$$

We recall another important theorem which is given in [20, Theorem 1.3].

Theorem 1.3.5. *The Mordell Weil group $MW(X)$ of the elliptic fibration on the surface X is isomorphic to the quotient $NS(X)/Tr_X$.*

Assume $f : X \longrightarrow \mathbb{P}^1$ admits an $n-$torsion section $\bar{s}$. If X is $K3$ surface, the section $\overline{s}$ induces an automorphism of the same order on X, defined as the identity on the base of the fibration and as a translation by the section on each fiber. This automorphism is a symplectic automorphism. In fact, the nowhere vanishing holomorphic two form of an elliptic $K3$ surface X is locally given by $f(\tau)(dx/y) \wedge d\tau$, where $f(\tau)$ is a nowhere vanishing holomorphic function. Let $dz = dx/y$. Then dz is a holomorphic form on each fiber E. The automorphism induced by the torsion section acts as the identity on τ, because it fixes the base of the elliptic fibration. Moreover the automorphism acts on a fiber E as a translation, so it fixes dz. In other words, $f(\tau)dz \wedge d\tau$ is fixed by the automorphism.

Chapter 2

Non-symplectic automorphisms of order 8

In this chapter we study non-symplectic automorphisms of order eight on $K3$ surfaces. We obtain a complete classification for the non-symplectic automorphisms of order 8 on a $K3$ surfaces, and we give several examples showing the existence of several cases in this classification

2.1 The fixed locus.

Let X be a $K3$ surface with a *non-symplectic* automorphism σ of order 8, this means that the action of σ^* on the vector space $H^{2,0}(X) = \mathbb{C}\omega_X$ of holomorphic two-forms is not trivial. More precisely we assume that $\sigma^*\omega_X = \zeta_8\omega_X$, where $\zeta_8 = e^{\frac{2\pi i}{8}}$ is a primitive 8th root of the unity. We call σ in this case a *(purely) non-symplectic* automorphism for simplicity we just call it *non-symplectic.*

We denote by $r_{\sigma^i}, l_{\sigma^i}, m_{\sigma^i}$ and m_1 for $i = 1, 2, 4$ the rank of the eigenspace of $(\sigma^i)^*$ in $H^2(X, \mathbb{C})$ relative to the eigenvalues $1, -1, i$ and ζ_8 respectively (clearly $m_{\sigma^4} = 0$). For simplicity we just write r, l, m for $i = 1$. We recall the invariant lattice:

$$S(\sigma^j) = \{x \in H^2(X, \mathbb{Z}) \,|\, (\sigma^j)^*(x) = x\},$$

and its orthogonal

$$T(\sigma^j) = S(\sigma^j)^{\perp} \cap H^2(X, \mathbb{Z}).$$

Observe easily that $\mathrm{rk}\ \mathrm{S}(\sigma^{\mathrm{i}}) = \mathrm{r}_{\sigma^{\mathrm{i}}}$, we have moreover that $S(\sigma^i) \subseteq \mathrm{Pic(X)}$ and $r_{\sigma^i} > 0$ for all $i = 1, 2, 4$ (see §1.2.2. Proposition 1.2.11). More precisely in the generic case we can assume that $\mathrm{Pic(X)} = \mathrm{S}(\sigma^4)$ as we have remarked in §1.2.2.

The following result will be useful later:

Lemma 2.1.1. *Let μ be a purely non-symplectic automorphism of finite order n, such that $Pic(X) = S(\mu)$. Then μ preserves each smooth rational curve in X (where it is either pointwise fixed by μ or it contains two isolated fixed points).*

Proof. Let R be a smooth rational curve in X. Since $S(\mu) = \mathrm{Pic}(X)$, the class of R in $\mathrm{Pic}(X)$ equals to the class of $\mu(R)$ (i.e $R \sim \mu(R)$ in Pic(X)). On the other hand, since by [12, (2.5.1)] $h^0(X, \mathcal{O}_X(R)) = \dim(\mathrm{H^0(X, \mathcal{O}_X(R))}) = \dim(\{\mathrm{D}$ effective divisor ; $D \sim R\}) = 1$, then there is only one effective divisor rationally equivalent to R which is R itself, so $\mu(R) = R$. Hence any smooth rational curve in X is invariant for μ. □

On the other hand, since $S(\sigma^i) \subseteq \mathrm{Pic}(X)$ for $i = 1, 2, 4$ the transcendental lattice satisfies that $T_X \subseteq T(\sigma^i)$ for $i = 1, 2, 4$. We write $T(\sigma) := T(\sigma^1)$. recall that the action of σ on T_X and $T(\sigma)$ is by primitive roots of the unity, see [8, Theorem 3.1 (c)], moreover the following proposition holds:

Proposition 2.1.2. *Let σ be a non-symplectic automorphism of order eight on a $K3$ surface X with $S(\sigma^4) \cong Pic(X)$. Then all of the eigenvalues of σ^* on $H^2(X, \mathbb{Z}) \otimes \mathbb{C}$ which are the primitives 8th roots of the unity have the same multiplicity m_1 (i.e. $\mathrm{rk}\ \mathrm{T_X} = 4\mathrm{m_1}$). Moreover, the following inequalities hold:*

$$m_1 \leq 5 \quad , \quad 1 \leq m + m_1 \leq 7$$

Proof. The first half of the statement is proved by Proposition 1.2.13 of Chapter 1. On the other hand, since X is a $K3$ surface then, rk $T_X \leq 22$ thus $4m_1 \leq 22$ and so $m_1 \leq 5$. To find the second inequality we use the relation $4(\sum_{C_i \subseteq \mathrm{Fix}(\sigma^2)}(1 - g(C_i)) = 2(10 - l_{\sigma^2} - m_{\sigma^2})$ of [2, Proposition 1], where $2m_{\sigma^2} = \mathrm{rkT}(\sigma^4) = 4m_1$ and $l_{\sigma^2} = 2m$. Hence:

$$\begin{aligned} 2(\sum_{C_i \subseteq \mathrm{Fix}(\sigma^2)} (1 - g(C_i))) &= 10 - l_{\sigma^2} - m_{\sigma^2} \\ &= 10 - 2m - 2m_1. \end{aligned}$$

So that:

$$\sum_{C_i \subseteq \mathrm{Fix}(\sigma^2)} (1 - g(C_i)) = 5 - (m + m_1).$$

But for [2, Theorem 0.1] we get:

$$2(\sum_{C_i \subseteq \mathrm{Fix}(\sigma^2)} (1 - g(C_i))) + 4 \geq 0,$$

Thus:

$$\sum_{C_i \subseteq \mathrm{Fix}(\sigma^2)} (1 - g(C_i)) \geq -2.$$

So that $-2 \leq (\sum_{C_i \subseteq \mathrm{Fix}(\sigma^2)}(1 - g(C_i)) = 5 - (m + m_1)$ hence $m + m_1 \leq 7$. □

Remark 2.1.3.

- *It is a straightforward computation to show that the invariants $r_{\sigma^i}, l_{\sigma^i}, m_{\sigma^i}$ and m_1 with $i = 1, 2, 4$ satisfy the following relations:*

$$\begin{aligned} r_{\sigma^2} &= r + l; & r_{\sigma^4} &= r + l + 2m; \\ l_{\sigma^2} &= 2m; & l_{\sigma^4} &= 4m_1; \\ 2m_{\sigma^2} &= 4m_1. \end{aligned}$$

- *As a direct consequence of the previous relations one can get immediately that the invariants l_{σ^2} and m_{σ^2} of σ^2 are even numbers.*

We denote by $\mathrm{Fix}(\sigma^{\mathrm{i}})$ for $i = 1, 2, 4$ the fixed locus of the automorphism σ^i such that:

$$\mathrm{Fix}(\sigma^{\mathrm{i}}) = \{\mathrm{x} \in \mathrm{X} \mid \sigma^{\mathrm{i}}(\mathrm{x}) = \mathrm{x}\},$$

we can find easily that $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\sigma^2) \subseteq \mathrm{Fix}(\sigma^4)$. In fact this simple result beside the facts in Remark 2.1.3 will be very useful later when we want to do the classification of the non-symplectic automorphisms of order 8. So to describe the fixed locus of order 8 non-symplectic automorphisms we start recalling the following result about non-symplectic involutions (see Theorem 1.2.17 in Ch 1 and also [11, §4]).

Theorem 2.1.4. *Let τ be a non-symplectic involution on a $K3$ surface X. The fixed locus of τ is either empty, the disjoint union of two elliptic curves or the disjoint union of a smooth curve of genus $g \geq 0$ and k smooth rational curves.*

Moreover, its fixed lattice $S(\tau) \subset \mathrm{Pic(X)}$ is a 2-elementary lattice with determinant 2^a such that:

- *$S(\tau) \cong U(2) \oplus E_8(2)$ iff the fixed locus of τ is empty;*
- *$S(\tau) \cong U \oplus E_8(2)$ iff τ fixes two elliptic curves;*
- *$2g = 22 - rkS(\tau) - a$ and $2k = rkS(\tau) - a$ otherwise.*

We denote by $N_{\sigma^i}, k_{\sigma^i}$ for $i = 1, 2, 4$ the number of isolated points and smooth rational curves in $\mathrm{Fix}(\sigma^{\mathrm{i}})$ (remark that N_{σ^4} is always equals to 0 since $\mathrm{Fix}(\sigma^4)$ does not contain isolated points as we have seen in Theorem 2.1.4). For simplicity we write just $N := N_{\sigma^1}, k := k_{\sigma^1}$. The fixed locus by σ is given by:

$$\mathrm{Fix}(\sigma) = C \cup R_1 \cup \ldots \cup R_k \cup \{p_1, \ldots, p_N\}$$

where C is a smooth curve of genus $g \geq 0$, R_i are smooth disjoint rational curves and p_i are isolated points. Observe that by [2] the fixed locus $\mathrm{Fix}(\sigma)$ can never contain two elliptic curves. One can see this also directly since as we will see in Proposition 2.1.8, $L(\sigma) \neq 0$.

Remark 2.1.5. *The non-symplectic automorphism σ of order 8 acts on a set of smooth rational curves of X either trivially (i.e. each smooth rational curve is σ-invariant or eventually pointwise fixed by σ) or it exchanges smooth rational curves two by two, or finally σ permutes four rational curves between them (in fact each curve in the set of permuted smooth rational curves by σ has stabilizer group in $\langle\sigma\rangle$ of order 2, hence its $\sigma-$ orbit has length 4).*

Lemma 2.1.6. *Four cyclic permuted smooth rational curves, by a non-symplectic automorphism σ of order 8 on a $K3$ surface X, are each σ^4-invariant, eventually pointwise fixed by σ^4.*

Proof. We can prove it simply as follows. Let $C_i\, ; i \in \{1, ..4\}$ be four smooth rational curves such that $\sigma(C_i) = C_{i+1}\, ; i = 1, 2, 3$ and $\sigma(C_4) = C_1$, and assume that C_1 is invariant by σ^4, then $\sigma^4(C_2) = \sigma^4(\sigma(C_1)) = \sigma(\sigma^4(C_1)) = \sigma(C_1) = C_2$. In particular if C_1 is pointwise fixed, then one proves in a similar way that C_2 is pointwise fixed. A similar proof holds also for C_3 and C_4, so we get the statement.

□

We denote by $2a$ the number of exchanged smooth rational curves by σ and fixed by σ^2, and by $4s$ the number of smooth rational curves cyclic permuted by σ and fixed by σ^4 (and clearly they are interchanged by σ^2 two by two).

Remark 2.1.7. *Let a_{σ^2} be the number of pairs of rational curves interchanged by σ^2 and fixed (that means pointwisely fixed) by σ^4 (so that the number of curves is $2a_{\sigma^2}$), then $2a_{\sigma^2} = 4s$ and so $a_{\sigma^2} \in 2\mathbb{Z}$.*

This remark will be very useful later in the study of the fixed locus of σ.

Proposition 2.1.8. *Let σ be a purely non-symplectic automorphism of order 8 on a $K3$ surface X. Then $\mathrm{Fix}(\sigma)$ is the disjoint union of smooth curves and $N \geq 2$ isolated points. Moreover, the following relations hold:*

$$n_{2,7} + n_{3,6} = 2 + 4\alpha,$$
$$n_{4,5} + n_{2,7} - n_{3,6} = 2 + 2\alpha,$$
$$N = 2 + r - l - 2\alpha.$$

Here we denote by $n_{i,j}$ the number of isolated points of type $P^{i,j}$ which are fixed by σ, and $\alpha = \sum_{C \subset \mathrm{Fix}(\sigma)} (1 - g(C))$.

Proof. Let σ be a purely non-symplectic automorphism of order 8, then $\sigma^*(\omega_X) = \zeta_8 \omega_X$ where $\zeta_8 = e^{\frac{2\pi i}{8}}$. The action of σ at a point in $\mathrm{Fix}(\sigma)$ can be locally diagonalized as follows (up to permutation of the coordinates, but this does not play any role in the next discussion):

$$A_{1,0} = \begin{pmatrix} \zeta_8 & 0 \\ 0 & 1 \end{pmatrix}, A_{2,7} = \begin{pmatrix} i & 0 \\ 0 & \zeta_8^7 \end{pmatrix}, A_{3,6} = \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & -i \end{pmatrix}, A_{4,5} = \begin{pmatrix} -1 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}.$$

In the first case the point belongs to a smooth fixed curve, while in the other cases it is an isolated fixed point. we say that an isolated point $x \in \mathrm{Fix}(\sigma)$ is of type $P^{i,j}$ if the local action at x is given by $A_{i,j}$. We denote by $n_{i,j}$ the number of isolated fixed points by σ of type $P^{i,j}$. The Lefschetz number of σ is :

$$\begin{aligned} L(\sigma) &= \sum_{j=0}^{2} (-1)^j (\mathrm{tr}(\sigma^* | \mathrm{H^j(X, \mathcal{O}_X)})) \\ &= (\mathrm{tr}(\sigma^* | \mathrm{H^0(X, \mathcal{O}_X)}) - (\mathrm{tr}(\sigma^* | \mathrm{H^1(X, \mathcal{O}_X)}) + (\mathrm{tr}(\sigma^* | \mathrm{H^2(X, \mathcal{O}_X)}). \end{aligned}$$

For $K3$ surfaces we have $h^1(X, \mathcal{O}_X) = 0$, hence we have to compute the action of σ^* only on $H^0(X, \mathcal{O}_X)$ and $H^2(X, \mathcal{O}_X)$. By Serre duality we get:

$$H^2(X, \mathcal{O}_X) \cong H^0(X, \Omega_X^2)^\nu \text{ where } \mathrm{H^0(X, \Omega_X^2) = \mathbb{C}\omega_X}.$$

Since $\sigma^*(\omega_X) = \zeta_8 \omega_X$ we have that σ^* acts by multiplication by $\overline{\zeta_8}$ on $H^0(X, \Omega_X^2)^\nu$. And since the action is trivial on $H^0(X, \mathcal{O}_X)$ we obtain:

$$\begin{aligned} L(\sigma) &= 1 - 0 + \overline{\zeta_8} \\ &= 1 + \zeta_8^7. \end{aligned}$$

We remark immediately that since $L(\sigma) \neq 0$ the fixed locus is never empty.

Now we assume that the fixed locus of σ is as follows:

$$\mathrm{Fix}(\sigma) = \mathrm{C} \cup \mathrm{R_1} \cup \ldots \cup \mathrm{R_k} \cup \{\mathrm{p_1}, \ldots, \mathrm{p_N}\}$$

where C is a smooth curve of genus $g \geq 0$, R_i are smooth disjoint rational curves and p_i are isolated points.

We also have that the Lefschetz number $L(\sigma)$ for a non-symplectic automorphism of order p is given by the following formula (see [24, Theorem 4.6]) :

$$L(\sigma) = \sum_j \alpha(p_j) + \sum_k b(D_k).$$

Such that

$$\alpha(p_j) = \frac{1}{\det(\mathrm{I} - \sigma^*|_{\mathrm{T_{p_j}}})} \quad , \quad b(D_k) = \frac{1 - g(D_k)}{1 - \zeta_p} - \frac{\zeta_p}{(1-\zeta_p)^2} D_k^2,$$

where p_j is an isolated fixed point, D_k is a fixed curve and T_{p_j} is the tangent space at p_j. Since X is a $K3$ surface then the self-intersection of D_k is equal to $D_k^2 = 2g(D_k) - 2$. Hence $L(\sigma) = \sum\limits_j \alpha(p_j) + \sum\limits_k \frac{(1+\zeta_p)(1-g(D_k))}{(1-\zeta_p)^2}$, and so for σ of order 8 we get :

$$L(\sigma) = \frac{n_{2,7}}{det(I - A_{2,7})} + \frac{n_{3,6}}{det(I - A_{3,6})} + \frac{n_{4,5}}{det(I - A_{4,5})} + \alpha \frac{(1+\zeta_8)}{(1-\zeta_8)^2} \ ,$$

$$L(\sigma) = \frac{n_{2,7}}{(1-i)(1-\zeta_8^7)} + \frac{n_{3,6}}{(1+i)(1-\zeta_8^3)} + \frac{n_{4,5}}{2(1-\zeta_8^5)} + \alpha \frac{(1+\zeta_8)}{(1-\zeta_8)^2} \ .$$

where $\alpha = \sum\limits_{C \subset \mathrm{Fix}(\sigma)} (1 - g(C))$.

We introduce the following notation:

$Q := 2(1-i)(1-\zeta_8^7)(1+i)(1-\zeta_8^3)(1-\zeta_8^5)(1-\zeta_8)^2$, $Q_1 := \frac{Q}{(1-i)(1-\zeta_8^7)}$,
$Q_2 := \frac{Q}{(1+i)(1-\zeta_8^3)}$, $Q_3 := \frac{Q}{2(1-\zeta_8^5)}$, $Q_4 := \frac{Q(1+\zeta_8)}{(1-\zeta_8)^2}$ and $Q_5 := Q(1+\zeta_8^7)$.

where by comparing the two expressions of the Lefschetz number $L(\sigma)$ we have:

$$n_{2,7}Q_1 + n_{3,6}Q_2 + n_{4,5}Q_3 + \alpha Q_4 = L(\sigma) = Q_5.$$

By computing Q, Q_i for $i \in \{1, ..., 5\}$, with $\zeta_8 = \sqrt{2}/2 + i\sqrt{2}/2$, one obtains that:

$$-4\sqrt{2}\ i\ n_{2,7} + 4(\sqrt{2}-2)in_{3,6} + 4(1-\sqrt{2})in_{4,5} + 8(1+\sqrt{2})i\alpha = -8\sqrt{2}\ i.$$

Hence

$$\begin{aligned} (-4n_{2,7} + 4n_{3,6} - 4n_{4,5} + 8\alpha)\sqrt{2}i &= -8\sqrt{2}i. \\ (-8n_{3,6} + 4n_{4,5} + 8\alpha)i &= 0. \end{aligned}$$

Since $n_{i,,j}, \alpha \in \mathbb{Z}$, we obtain the two following relations :

$$\begin{aligned} n_{2,7} - n_{3,6} + n_{4,5} - 2\alpha &= 2. \\ -2n_{3,6} + n_{4,5} + 2\alpha &= 0. \end{aligned}$$

and so we get:

$$\begin{cases} n_{2,7} + n_{3,6} = 2 + 4\alpha. \\ n_{4,5} + n_{2,7} - n_{3,6} = 2 + 2\alpha . \end{cases} \qquad (**)$$

In particular this implies that the automorphism σ fixes at least 2 isolated points. In fact if $\alpha \geq 0$ then by the first relation of (**) we get $n_{2,7} + n_{3,6} \geq 2$. Otherwise, $\alpha \leq -1$ and so $n_{2,7} + n_{3,6} \leq -2$ by the first relation again which is not possible.

We consider now the topological Lefschetz fixed point formula:

$$\begin{aligned} \chi(\mathrm{Fix}(\sigma)) &= \sum_{j=0}^{4} (-1)^j \mathrm{tr}(\sigma^* | \mathrm{H^j}(\mathrm{X}, \mathbb{R})) \\ &= 2 + \mathrm{tr}(\sigma^* | \mathrm{H}^2(\mathrm{X}, \mathbb{R})) \\ &= 2 + r(1) + l(-1) + m(i - i) + m_1(\zeta_8 + \zeta_8^3 + \zeta_8^5 + \zeta_8^7) \\ &= 2 + r - l. \end{aligned}$$

Where $\mathrm{tr}(\sigma^*|\mathrm{H}^0(\mathrm{X}, \mathbb{R})) = \mathrm{tr}(\sigma^*|\mathrm{H}^4(\mathrm{X}, \mathbb{R})) = 1,\ \mathrm{tr}(\sigma^*|\mathrm{H}^1(\mathrm{X}, \mathbb{R})) = \mathrm{tr}(\sigma^*|\mathrm{H}^3(\mathrm{X}, \mathbb{R})) = 0$, and m is the multiplicity of $\mp i$, m_1 is the multiplicity of the 4 eigenvalue ζ_8 (see Proposition 2.1.2).

On the other hand, the Euler-Poincaré charactaristic $\chi(\mathrm{Fix}(\sigma))$ is also given by:

$$\chi(\mathrm{Fix}(\sigma)) = \sum_{\mathrm{C_i} \subset \mathrm{Fix}(\sigma)} \chi(\mathrm{C_i}) + \sum_{\mathrm{P_i}} \chi(\mathrm{P_i}) . \qquad (\mathrm{I})$$

Where

$$\begin{aligned} \chi(C) &:= h^0(C, \mathbb{Z}) - h^1(C, \mathbb{Z}) + h^2(C, \mathbb{Z}) \\ &= 1 - 2g(C) + 1 \\ &= 2 - 2g(C). \end{aligned}$$

Substituting in the relation (I) we get the following equality:

$$\begin{aligned} \chi(\mathrm{Fix}(\sigma)) &= \sum_{C_i \subset \mathrm{Fix}(\sigma)} 2(1 - g(C_i)) + N \\ &= 2\alpha + n_{2,7} + n_{3,6} + n_{4,5}. \end{aligned}$$

Comparing the two formulas for $\chi(\mathrm{Fix}(\sigma))$ we obtain the relation :

$$2\alpha + n_{2,7} + n_{3,6} + n_{4,5} = 2 + r - l .$$

Comparing with (**) we get:

$$\begin{aligned} n_{2,7} + n_{4,5} &= \frac{(4 + r - l)}{2} ; \\ n_{2,7} + 3n_{3,6} &= 2 + r - l ; \\ 2\alpha &= 2 - N + r - l . \end{aligned}$$

□

Remark 2.1.9. *The isolated fixed points, by a non-symplectic order eight automorphism σ of type $P^{2,7}$ and $P^{3,6}$ are also isolated points in* $\mathrm{Fix}(\sigma^2)$. *The points of type $P^{4,5}$ in* $\mathrm{Fix}(\sigma)$ *are contained in a smooth fixed curve by σ^2. In fact the action of σ^2 at a point of type $P^{4,5}$ is given by the matrix* $\begin{pmatrix} 1 & 0 \\ 0 & \zeta_8^2 \end{pmatrix}$ *which implies that this point belongs to a smooth curve in* $\mathrm{Fix}(\sigma^2)$.

From now on we denote by n_i instead of $n_{i,j}$ the number of isolated fixed points by σ of type $P^{i,j}$, where $i = 2, 3, 4$ and $i + j = 9$.

We recall now Lemma 1.2.15 and the following useful remark which is a direct application of the Remark 1.2.16 when the order of a non-symplectic automorphism is $n = 8$ (see also e.g. [2, Lemma 4]):

Lemma 2.1.10. *Let $T = \sum_i R_i$ be a tree of smooth rational curves on a $K3$ surface X such that each R_i is invariant under the action of a purely non-symplectic automorphism σ of order q. Then, the points of intersection of the rational curves R_i are fixed by σ and the action at one fixed point determines the action on the whole tree.*

Remark 2.1.11. *In the case of an automorphism of order 8, with the assumption of Lemma 2.1.10, the local actions at the intersection points of the curves R_i appear in the following order (we give only the exponents of ζ_8 in the matrix of the local action):*

$$\ldots, (0,1), (7,2), (6,3), (5,4), (4,5), (3,6), (2,7), (1,0), \ldots$$

Assuming that $T = R$ consists of only one rational curve, which is not pointwise fixed, one get immediately that σ has either one fixed point of type $P^{2,7}$ and another one of type $P^{3,6}$ or two fixed points of type $P^{4,5}$.

2.2 Elliptic fibrations.

Proposition 2.2.1. *Let σ be a purely non-symplectic automorphism of order 8 on a $K3$ surface X such that* $\mathrm{Pic(X)} = \mathrm{S}(\sigma^4) \cong \mathrm{U} \oplus \mathrm{L}$ *and σ^4 fixes a curve C of genus $g > 1$. Then X carries a jacobian elliptic fibration $\pi : X \longrightarrow \mathbb{P}^1$ whose fibers are σ^4-invariant . Moreover:*

- *If L is isomorphic to a direct sum of root lattices of types* $\mathrm{A_n}, \mathrm{D_{4+n}}, \mathrm{E_7}$ *or* $\mathrm{E_8}$ *then π has reducible fibers described by L and a unique section* $E \subset \mathrm{Fix}(\sigma^4)$. *The fibration π is σ-invariant if $g > 4$. Finally the genus of a curve in* $\mathrm{Fix}(\sigma)$ *(if it exists) is equal to 2.*

- *If L is not isomorphic to a direct sum of root lattices, then π has two sections $E, E' \subset$* $\mathrm{Fix}(\sigma^4)$ *and C intersects each fiber in two points. The fibration π is $\sigma-$invariant if $g > 2$. Finally the genus of a curve in* $\mathrm{Fix}(\sigma)$ *(if it exists) is 2 and we get $k \geq 1$ in this case.*

Proof. The first half of the statement follows from [5, Lemma 2.1, 2.2]. If σ^4 fixes a curve C of genus $g > 1$, then this curve is transversal to the fibers of π. This implies that σ^4 preserves each fiber of π and acts as an involution on it with four fixed points. If

L is a root lattice, then by [6, Theorem 6.3] we have that the Mordell-Weill group of π is $MW(\pi) \cong \mathrm{Pic(X)}/\mathcal{T}$ where $\mathcal{T}$ denotes the subgroup of Pic(X) generated by the zero section and fiber components. Hence since L is a root lattice and $\mathrm{Pic(X)} \cong \mathrm{U} \oplus \mathrm{L}$ we have that $MW(\mathcal{T})$ is trivial, thus π has a unique section E. Since σ^4 preserves each fiber of π, we have $E \subset \mathrm{Fix}(\sigma^4)$. Eventually this implies that C intersects each fiber in three points with multiplicity (since σ^4 acts as an involution on a generic fiber of π with 4 fixed points on each one of them) and one fixed point for the action of σ^4 is contained in the section E.

Now we prove that π is $\sigma-$invariant if $g > 4$. Let f be the class of a fiber of π and s be the class of the curve C. The automorphism σ preserves the curve C, and we have that $C \cap E = 0$ since the fixed curves for σ^4 can not intersect. Assume that $f \neq \sigma^*(f)$ then they intersect in at least 2 points. In fact if $f.\sigma^*(f) = 1$ then this is a fixed point on f and so either C is fixed by σ which is not possible (since $f.\sigma^*(f) \geq 3$ in this case, where C intersects each fiber in 3 points), or E is fixed by σ. This is not possible too, since otherwise the action of σ on the basis of the fibration would be the identity and so $f = \sigma^*(f)$ which contradicts the assumption $f \neq \sigma^*(f)$. Now applying [2, Lemma 5] we find that:

$$2g - 2 = s^2 \leq \frac{2(s.f)^2}{f.\sigma^*(f) + 1} \leq \frac{2.9}{3} \leq 6,$$

which implies that $g = g(C) \leq 4$ if π is not $\sigma-$invariant. Hence π is σ-invariant if $g > 4$. Now we suppose that σ fixes the curve C, then by [2, Theorem 2.1] we get $g \leq 2$ and π is not invariant by σ (where C is contained automatically in $\mathrm{Fix}(\sigma^2)$).

If L is not a root lattice, then the Mordell-Weil group of π is not trivial, so that π has at least two sections $E', E \subset \mathrm{Fix}(\sigma^4)$. Since σ^4 fixes 4 points on each generic fiber, we have that $s.f \leq 2$. If $s.f = 1$ (i.e there are three sections of π fixed by σ^4) then $g \leq 1$ by [2, Lemma 5] that is not possible. Hence C intersects each fiber in two points. In this case $2 < g \leq 5$ if π is σ-invariant by [2, Lemma 5] such that σ does not have any fixed point on a generic fiber of π, and $g \leq 2$ by the same [2, Lemma 5] otherwise.
If σ fixes a curve C, then $f.\sigma^*(f) \geq 2$ since σ fixes two points on each fiber and so it cannot preserve a generic fiber of π (since otherwise each fiber would admit an automorphism of order 8, hence σ would be a translation which is impossible too). Hence π is not invariant by σ which implies that $g \leq 2$ by [2, Lemma 5]. Moreover, σ fixes at least one smooth rational curve. In fact by Proposition 2.1.8 we have that $n_2 + n_3 = 2 + 4\alpha$ where $\alpha = k + 1 - g(C) = k - 1$ since $g(C) = 2$. So that $k > 0$ (where $n_2 + n_3 = 4k - 2$ is a positive number).

□

2.3 The case of an invariant elliptic curve.

In this section we suppose that σ^4 fixes an elliptic curve C. Thus the $K3$ surface X carries an elliptic fibration $\pi_C : X \longrightarrow \mathbb{P}^1$ having C as a smooth fiber. Observe that π_C is invariant by σ^i ; $i = 1, 2, 4$ (since σ^i preserves C which is a fiber of π_C) and all curves fixed by σ^i are contained in the fibers of π_C, that because they are disjoint with C and the action on the base of σ is non-trivial. In fact if the action would be trivial then a smooth fiber would have an automorphism of order 8. An elliptic curve can admit only automorphisms of order 2,4,6 (different from a translation), so that this automorphism should be induced by a translation by a point of order 8 on the generic fiber. But then σ would be a symplectic automorphism, which contradicts our assumption on σ. Here we distinguish the following three cases:

- The elliptic curve C is fixed by σ.
- The elliptic curve C is fixed by σ^2 but is not contained in $\mathrm{Fix}(\sigma)$.
- The elliptic curve C is only fixed by σ^4.

Lemma 2.3.1. *If X carries a $\sigma-$invariant elliptic fibration, such that σ^4 fixes an irreducible smooth fiber C of this fibration, then σ acts with order 8 on the basis of the fibration and fixes two points on it.*

Proof. Let $\pi_C : X \longrightarrow \mathbb{P}^1$ be a $\sigma-$invariant elliptic fibration having C as a smooth fiber and such that C is fixed by the involution σ^4. Observe that σ^2 (respectively σ^4) is not the identity on the base of π_C, since otherwise it would act as the identity on the tangent space at a point of C, contradicting the fact that σ^2 (respectively σ^4) is purely non-symplectic. Hence σ acts as an order *eight* automorphism on $\mathbb{P}^1$ and has two fixed points on it corresponding to C and another fibre $C^{'}$. □

2.3.1 The fixed locus $\mathrm{Fix}(\sigma)$ contains an elliptic curve.

Here we assume that σ fixes an elliptic curve C, and so it will be also fixed by a non-symplectic order four automorphism σ^2.

Theorem 2.3.2. *Let σ be a purely non-symplectic order eight automorphism on a $K3$ surface X with $\mathrm{Pic(X)} = \mathrm{S}(\sigma^4)$ and $\pi_C : X \longrightarrow \mathbb{P}^1$ be an elliptic fibration with a smooth fiber $C \subset \mathrm{Fix}(\sigma)$. Then σ preserves π_C and acts on its base as an order eight automorphism with two fixed points corresponding to the fiber C and a fiber $C^{'}$ which is either smooth or of Kodaira type IV^*. The corresponding invariants of σ are given in Table 2.1 .*

m_1	m	r	l	(n_2, n_3, n_4)	N	k	A	type of $C^{'}$
3	2	3	3	$(2, 0, 0)$	2	0	0	I_0
2	2	6	4	$(1, 1, 2)$	4	0	1	IV^*

Table 2.1: The case g=1, $C \subseteq \mathrm{Fix}(\sigma)$.

Proof. By Lemma 2.3.1 we get that σ has order 8 on $\mathbb{P}^1$ with two fixed points on it corresponding to C and another fiber $C^{'}$, which is either smooth elliptic or reducible such that it contains all fixed rational curves by σ^4.

If $C^{'}$ is irreducible, then $\alpha = k = 0$ thus $N = 2 + r - l$ and $N = n_4 + 2 = 2n_3 + 2$ by Proposition 2.1.8. We observe that $n_4 = 0$, since otherwise σ^2 would fix a curve transversal to $C^{'}$ that should meet C (where $C^{'}$ is not fixed by σ^2 since it does not fix two elliptic curves by [2, Proposition 1]). But C is also fixed by σ^2, so this is not possible. Hence we get that $N = 2$ and $n_3 = n_4 = 0$ thus $n_2 = 2$, which implies that $C^{'}$ is smooth elliptic and σ has order four on it (where it fixes two isolated points on $C^{'}$). Using the fact that

$(r_{\sigma^2}, l_{\sigma^2}, m_{\sigma^2}) = (6, 4, 6)$ (see [2, Theorem 3.1]) we get immediately that $m_1 = 3, m = 2$. Moreover we have that $r - l = 0$ and $r + l = 6$. So that $r = l = 3$.

We now assume that C' is reducible and we classify the possible $Kodaira$ types for it. Since the fibration admits also an automorphism of order four σ^2, by [2, Theorem 3.1] the possible types of C' are: $I_4, IV^*, I_8, I_{12}, I_{16}$. Observe that the cases C' is of type I_8 and I_{16} with $a_{\sigma^2} = 1, 3$ respectively contradict Remark 2.1.7. On the other hand, the cases when C' is of type I_4 and I_{12} can be excluded since $m_{\sigma^2} = 2m_1$ and $l_{\sigma^2} = 2m$ are even numbers.

If $C' = IV^*$, then the case with $a_{\sigma^2} = 1$ is not possible by Remark 2.1.7 again. We are left with the case $C' = IV^*$ and $(a_{\sigma^2}, k_{\sigma^2}, N_{\sigma^2}) = (0, 1, 6)$. Observe that $n_2 = n_3$ (see Remark 2.1.11, where the fixed points by σ are contained in smooth rational curves) so that we get $2n_2 = 2 + 4\alpha$ by Proposition 2.1.8. If $k = 1$, then $n_2 = n_3 = 3$ and $n_4 = 0$ (since $k_{\sigma^2} = k = 1$), but that contradicts the fact that $n_4 > 0$ by the second equality of Proposition 2.1.8. Thus $k = 0$ and so $n_2 = n_3 = 1$, $n_4 = 2$ and σ acts as a reflection on the fiber IV^*.
By the same previous argument one computes easily the values of m_1, m, r and l as in Table 2.1. □

2.3.2 The fixed locus $\mathrm{Fix}(\sigma^2)$ contains an elliptic curve.

We assume that $C \subset \mathrm{Fix}(\sigma^2)$ is not fixed by σ, since we have already discussed this case in Theorem 2.3.2. Observe however that one can easily show that σ preserves C. We recall first some notations. Let N' be the number of isolated fixed points of σ on a curve $C \subseteq \mathrm{Fix}(\sigma^4)$ of genus $g \geq 1$, and let $2A$ be the number of smooth rational curves interchanged by σ and invariants (but not pointwise fixed) by σ^2. We remark that $N_{\sigma^2} = (n_2 + n_3) + 4A$ and the fixed points on C are of type $P^{4,5}$ in this case.

Theorem 2.3.3. *Let σ be a purely non-symplectic automorphism of order eight on a $K3$ surface X with $\mathrm{Pic(X)} = \mathrm{S}(\sigma^4)$ and C be an elliptic curve in $\mathrm{Fix}(\sigma^2)$ and assume there is no elliptic fixed curve by σ. Then the corresponding invariants of σ are given in Table 2.2.*

m_1	m	r	l	N	(n_2, n_3, n_4)	N'	k	a	A	type of C'
3	2	3	3	2	(2,0,0)	0	0	0	0	I_0
3	2	5	1	6	(0,2,4)	4	0	0	0	I_0
2	2	6	4	4	(1,1,2)	0	0	0	1	IV^*
2	2	10	0	10	(3,3,4)	4	1	0	0	IV^*

Table 2.2: The case g=1 , $\mathrm{Fix}(\sigma^2) \supseteq C \nsubseteq \mathrm{Fix}(\sigma)$.

Proof. Since σ preserves C, then there is a $\sigma-$invariant elliptic fibration $\pi_C : X \longrightarrow \mathbb{P}^1$ with fiber C. Observe that by Lemma 2.3.1 the automorphism σ has order eight on the basis of π_C and it has two fixed points on $\mathbb{P}^1$, corresponding to the fiber C and a fiber C' of π_C. This implies that all rational curves fixed by σ are contained in C'. Observe that σ acts on C either as an involution with four fixed points or as a translation. By the same argument as in Theorem 2.3.2 we have that C' is either a smooth elliptic curve or of type IV^*.

If C' is irreducible, then $\alpha = k = 0$ thus $N = 2 + r - l$ and $N = n_4 + 2 = 2n_3 + 2$ by Proposition 2.1.8. Since $\mathrm{Fix}(\sigma^2)$ can not contain two elliptic curves (observe that C' must be smooth), and since σ acts on C either as an involution with four fixed points of type $P^{4,5}$ or as a translation with zero fixed point, we have n_4 either equal to 0 or to 4 and $(n_2, n_3) =$ (2,0) or (0,2) by Proposition 2.1.8. In these two cases C' is smooth elliptic and σ acts on it as an order four automorphism with two fixed points.

If C' is of type IV*, then the central component of multiplicity 3 is invariant for σ. If the central component is fixed by σ, then $k = 1$, $A = a = 0$ and by Lemma 2.1.10 we have that $n_2 = n_3 = 3$. By Proposition 2.1.8 we get $n_4 = 2 + 2k = 4$ thus σ acts as an involution on C and $N' = 4$. Otherwise the two branches of the fiber C' are exchanged by σ thus $k = 0$ and $A = 1$. On other hand, the central component of C' has two isolated fixed points of type $P^{4,5}$. By applying Lemma 2.1.10 we obtain that $n_2 = n_3 = 1$ and so $n_4 = 2$ by Proposition 2.1.8. In this case σ acts as a translation on C. □

2.3.3 The fixed locus $\mathrm{Fix}(\sigma^4)$ contains an elliptic curve.

We assume that $\mathrm{Fix}(\sigma)$ and $\mathrm{Fix}(\sigma^2)$ do not contain an elliptic curve. Observe that the fixed points by σ on C are of type $P^{2,7}$ and $P^{3,6}$.

Theorem 2.3.4. *Let σ be a purely non-sympletic automorphism of order eight such that $\mathrm{Fix}(\sigma^4)$ contains an elliptic curve C and $\sigma^i, i = 1, 2$ does not fix an elliptic curve. Then we are in one of the cases appearing in Table 2.3.*

m_1	m	r	l	N	(n_2, n_3, n_4)	N'	k	a	type of C'
3	2	3	3	2	(2,0,0)	2	0	0	I_0
2	1	10	2	8	$(4, 2, 2)$	2	1	0	I_8
2	1	8	4	6	(0,2,4)	2	0	0	I_8
2	1	6	6	2	(2,0,0)	2	0	1	I_8
2	3	4	4	2	(2,0,0)	2	0	$0(s = 1)$	I_8
1	0	17	1	14	(6,4,4)	2	2	0	I_{16}
1	0	11	7	6	(0,2,4)	2	0	1	I_{16}
1	0	9	9	2	(2,0,0)	2	0	2	I_{16}
1	4	5	5	2	(2,0,0)	2	0	$0(s = 2)$	I_{16}
2	2	6	4	4	(1,1,2)	0	0	0	IV*

Table 2.3: The case $g = 1, \mathrm{Fix}(\sigma^4) \supseteq \mathrm{C} \not\subseteq \mathrm{Fix}(\sigma^2)$.

Proof. As in the proof of theorems 2.3.2 and 2.3.3 the $K3$ surface X admits a $\sigma-$invariant elliptic fibration. By Lemma 2.3.1 the automorphism σ acts with order 8 on the basis of the fibration with two fixed points. Let C and C' be the fibers over these two points.

If C' is irreducible, then σ acts on it either as an order four automorphism with two fixed points (of type $P^{2,7}, P^{3,6}$) or as a translation. Observe that $n_4 = 0$ in this case. Since $k = 0$ and $n_4 = 0$ (where π_C does not have a fixed section since the curves in $\mathrm{Fix}(\sigma^4)$ do not intersect) we have by Proposition 2.1.8 that $(n_2, n_3) = (2, 0)$ and $r - l + 2 = 2$ so $r = l$. Hence σ acts as a translation on one of this two elliptic curves and as an automorphism of order 4 on the other fiber. In this case σ^2 only has isolated fixed points so that we are in the case $g = 1$ and $(m_{\sigma^2}, r_{\sigma^2}) = (6, 6)$ of Table 6 of [2].

If C' is reducible, then we will distinguish the following two cases:

- $k_{\sigma^2} = 0$:
  In this case we have that $k = n_4 = 0$ thus $N = 2$ and $(n_2, n_3) = (2, 0)$ by Proposition 2.1.8. Observe that these two fixed points belong to the curve C (they can not be contained in a rational curve by Remark 2.1.11) and so the four fixed points by σ^2 are contained in C (here $N_{\sigma^2} = 4$ by [2, Proposition 1]). On the other hand, we have $k_{\sigma^2} = k_\sigma = 0$, this means that σ and σ^2 do not fix any point in C' which excludes the *Kodaira* types $III, IV, I_b^*, II^*, III^*$ and IV^* of C' (where otherwise we get a contradiction with the fact that π is invariant by σ).

  Assume that now C' is of type I_M ; $M \geq 4$. Since $k_{\sigma^4} = 2a_{\sigma^2}$ in this case (where $k_{\sigma^2} = 0$ and the isolated fixed points by σ^2 are contained in the smooth fiber C) and all fixed curves by σ^4 are contained in C', we get $M = 2k_{\sigma^4} = 4a_{\sigma^2}$ (in fact all components of C' are preserved by σ^4 since Pic(X) = S(σ^4) and a component which is not fixed intersects two fixed ones). Hence C' is of type $I_{4a_{\sigma^2}}$. By Lemma 2 we have $a_{\sigma^2} \in 2\mathbb{Z}$ so that C' is of type I_8 with $2a_{\sigma^2} = 4$ or of type I_{16} with $2a_{\sigma^2} = 8$. That corresponds respectively to the case $(r_{\sigma^2}, m_{\sigma^2}) = (8, 4), (10, 2)$ in [2, Table 6] . Assume that now C' is of type I_M ; $M \geq 4$. Since $k_{\sigma^4} = 2a_{\sigma^2}$ in this case (where $k = n_4 = 0$) and all fixed curves by σ^4 are contained in C', thus $M = 2k_{\sigma^4} = 4a_{\sigma^2}$. In fact all components of C' are preserved by σ^4 since Pic(X) = S(σ^4) and a component which is not fixed intersects two fixed ones. Hence C' is of type $I_{4a_{\sigma^2}}$. By Remark 2.1.7 we have $a_{\sigma^2} \in 2\mathbb{Z}$ so that C' is of type I_8 with $2a_{\sigma^2} = 4$ or of type I_{16} with $2a_{\sigma^2} = 8$. That corresponds to the case $(r_{\sigma^2}, m_{\sigma^2}) = (8, 4)$ respectively $(10, 2)$ in Table 6 of [2].

- $k_{\sigma^2} \neq 0$:
  Here we will study the cases when $l_{\sigma^2} = 0$ and $l_{\sigma^2} \neq 0$ (i.e. the action of σ^2 is trivial or non trivial).

  (i) $l_{\sigma^2} \neq 0$:
  By [2, Theorem 8.4] and since Fix(σ) $\subseteq$ Fix(σ^2) and $m_{\sigma^2} = 2m_1$ is an even number, we get that C' is either of Kodaira type IV^* and σ^2 acts on C as a translation, or it is of type I_8 such that $k_{\sigma^2} = 2$ and σ^2 acts as an involution on C. If C' is of type IV^*, then the central component of multiplicity 3 is invariant (not necessarily pointwise fixed) by σ and pointwise fixed by σ^2: in fact since C' is preserved by $\sigma^i, i = 1, 2, 4$ thus either σ preserves each component of IV^* or it exchanges the two branches of IV^*. In this two cases σ^2 preserves each component of C' and it fixes the central component of multiplicity 3 since it has at least three fixed points by σ^2 (here we have $k_{\sigma^2} = 1$ and so $k \leq 1$). We have then that either $k = 1$ when σ preserves each component of C', hence $a = A = 0$ and $(n_2, n_3, n_4) = (3, 3, 4)$ by Proposition 2.1.8. On the other hand, by applying Lemma 2.1.10 one get that the fiber C' contains only 3 fixed points of type $P^{2,7}$ and 3 fixed points of type $P^{3,6}$. So that the four fixed points by σ of type $P^{4,5}$ are contained in the smooth fiber C. That is not possible since otherwise σ would act on C as an involution which contradicts our assumption. Or $k = 0$ such that the automorphism σ exchanges the two branches of C'. By applying Lemma 2.1.10 we get that the fiber C' contains two

fixed points of type $P^{4,5}$ (on the central component of multiplicity 3) and one point of type $P^{2,7}$ and another one of type $P^{3,6}$. In this case σ acts on C as a translation where $(n_2, n_3, n_4) = (1, 1, 2)$ by Proposition 2.1.8.

Now assume that C' is of type I_8, thus it contains at most two fixed curves by σ since $k_{\sigma^2} = 2$. If $k = 2$ then $n_4 = 0$ and so $N = 10$ by Proposition 2.1.8. Hence σ fixes 6 points on C. It is impossible since C has at most 2 fixed points by σ on it. If $k = 1$, then $n_4 = 2$ where $k_{\sigma^2} = k + n_4/2 + 2a$. Applying Lemma 2.1.10 on C' we get that it has 3 points of type $P^{2,7}$, 2 points of type $P^{4,5}$ and one point of type $P^{3,6}$. By Proposition 2.1.8 we have that $N = 8$ and so C has two fixed points by σ (of type $P^{2,7}$ and $P^{3,6}$ respectively). Thus σ acts as an automorphism of order 4 on C. Finally if $k = 0$, then since $k_{\sigma^2} = 2 = n_4/2 + 2a$ we get $n_4 = 0, 4$ and $a = 1, 0$. If $n_4 = 0$ then $N = 2$, $(n_2, n_3) = (2, 0)$ and $r = l$ by Proposition 2.1.8. Thus σ acts on C' as a rotation with no fixed point on it. If $n_4 = 4$ and $a = 0$, then σ acts as a reflection on C'. Such that $(n_2, n_3) = (0, 2)$ and $r - l = 4$ by Proposition 2.1.8. Observe that σ acts on C as an automorphism of order 4 in this case (when $k = 0$).

(ii) $l_{\sigma^2} = 0$:

By equality (1) in [2, Theorem 8.1] we get that $1 - 2a_{\sigma^2} = m_{\sigma^2} + 1 - N'_{\sigma^2}/2$ (where N'_{σ^2} is the number of fixed points by σ^2 on C). Thus $m_{\sigma^2} = N'_{\sigma^2}/2 - 2a_{\sigma^2}$. Since $N'_{\sigma^2} =$ 0 or 4 and $a_{\sigma^2} \in 2\mathbb{Z}$ by Remark 2.1.7, then $(m_{\sigma^2}, N'_{\sigma^2}, a_{\sigma^2}) = (2, 4, 0)$ and σ^2 acts on C as an involution. By [2, Proposition 1] one can find that $2k_{\sigma^2} = 10 - m_{\sigma^2} = 8$ and so $k_{\sigma^2} = 4$. On the other hand, by the same previous argument and since $k_{\sigma^2} = 4$ we get that C' is a reducible fiber containing four fixed rational curves by σ^2. So by checking all the possibilities of Kodaira types for fibers one gets that C' is of type I_M such that $M \geq 8$. In fact all components of I_M are preserved by σ^4 and a component which is not fixed by σ^4 intersects two fixed ones, so that $M = 2k_{\sigma^4}$. On the other hand, we have that $k_{\sigma^4} = k_{\sigma^2} + (N_{\sigma^2} - N'_{\sigma^2})/2 + 2a_{\sigma^2}$, since $(N'_{\sigma^2}, a_{\sigma^2}) = (4, 0)$ and $N_{\sigma^2} = 2k_{\sigma^2} + 4$ by [2, Proposition 1] we get $k_{\sigma^4} = 2k_{\sigma^2}$. So that $M = 2k_{\sigma^4} = 2(k_{\sigma^2}) = 4k_{\sigma^2}$. Hence C' is of type I_{16} in this case (where $k_{\sigma^2} = 4$). If $k \neq 0$, then σ would preserve each components of C'. Thus $a = 0$, $k = 2, (n_2, n_3, n_4) = (6, 4, 4)$ by Lemma 2.1.10. Otherwise $k = 0$ thus either $n_4 = 4, a = 1$ so σ acts as a reflection on C', or $n_4 = 0, a = 2$ then σ acts on C' as a rotation with no fixed point on it.

Finally, to find the invariants r, l and m_1 of σ in all the previous cases we use the facts that $r + l = r_{\sigma^2}$, $2m_1 = m_{\sigma^2}$ and the relation $N = 2 + r - l - 2k$ in Proposition 2.1.8. □

2.4 The case when σ^2 acts as the identity on the Picard group.

In this section we will assume that $m = l_{\sigma^2} = 0$ (i.e. σ^2 acts as the identity on $S(\sigma^4) \cong$ Pic(X)). We recall first some notations. Let g_{σ^i} for $i = 1, 2, 4$ denotes the genus of the curve $C \subset \mathrm{Fix}(\sigma^i)$, $2h$ the number of interchanged points by σ on the curve C, and we denote by N'_{σ^i} for $i = 1, 2$ to be the number of isolated fixed points by σ^i on C (for simplicity we will just write N' for $i = 1$).

Theorem 2.4.1. *Let σ be a purely non-symplectic automorphism of order 8 on a $K3$ surface X such that σ^2 acts trivially on* Pic(X) $\cong$ S(σ^4)*. Then the invariants of the fixed locus of σ, the lattice* S(σ^4) = S(σ^2) *appear in Table 2.4 see also Table 2.3 for $g = 1$.*

r	l	m_1	(n_2,n_3,n_4)	N'	h	k	a	g_{σ^4}	$S(\sigma^4)$
1	1	5	(2,0,0)	2	1	0	0	9	$U(2)$
6	0	4	(5,1,0)	4	0	1	0	7	$U\oplus D_4$
4	2	4	(1,1,2)	2	2	0	0	6	$U(2)\oplus D_4$
6	0	4	(5,1,0)	6	0	1	0		
7	3	3	(0,2,4)	2	1	0	0	5	$U(2)\oplus E_8$
9	1	3	(2,0,0)	2	1	0	1		
9	1	3	(4,2,2)	2	1	1	0		
9	1	3	(4,2,2)	4	1	1	0	4	$U\oplus D_4^{\oplus 2}$
7	3	3	(0,2,4)	2	3	0	0	3	$U(2)\oplus D_4^{\oplus 2}$
5	5	3	(2,0,0)	2	3	0	1		
8	6	2	(1,1,2)	0	2	0	1	3	$U\oplus E_8\oplus D_4$
12	2	2	(3,3,4)	(4-0)	(0-2)	1	0		
10	4	2	(5,1,0)	4	0	1	1		
14	0	2	(7,3,2)	4	0	2	0		
8	6	2	(1,1,2)	2	2	0	1	2	$U(2)\oplus E_8\oplus D_4$
12	2	2	(3,3,4)	2	2	1	0		

Table 2.4: The case $m=0, g_{\sigma^4}>1$.

Proof. Since σ^2 acts trivially on Pic(X) $\cong$ S(σ^4), then $2m=l_{\sigma^2}=0$ and by [2, Proposition 5] we have $4s=2a_{\sigma^2}=0$. So that the invariants of the fixed locus of σ^2 are one of the cases appearing in Table 5 of [2, Theorem 6.1] that is:

m_{σ^2}	r_{σ^2}	N_1	N'_{σ^2}	k_{σ^2}	g_{σ_4}	$S(\sigma^4)$
10	2	2	2	0	10	U
	2	0	4	0	9	$U(2)$
8	6	2	4	1	7	$U\oplus D_4$
	6	0	6	1	6	$U(2)\oplus D_4$
6	10	6	2	2	6	$U\oplus E_8$
	10	4	4	2	5	$U(2)\oplus E_8$
	10	2	6	2	4	$U\oplus D_4^{\oplus 2}$
	10	0	8	2	3	$U(2)\oplus D_4^{\oplus 2}$
4	14	6	4	3	3	$U\oplus D_4\oplus E_8$
	14	4	6	3	2	$U(2)\oplus D_4\oplus E_8$
2	18	10	2	4	2	$U\oplus E_8^{\oplus 2}$
	18	8	4	4	1	$U(2)\oplus E_8^{\oplus 2}$

Here $N_1=N_{\sigma^2}-N'_{\sigma^2}$ denotes the number of isolated fixed points of σ^2 contained in smooth rational curves.

Observe that the fixed points by σ on C are of type $P^{2,7}$ and $P^{3,6}$ since σ acts on C as an automorphism of order four. By Riemann-Hurwitz formula applied to the automorphism σ on C we have that:

$$2g(C)-2=\deg\,(\sigma_{|_C})(2g(D)-2)+\deg R,$$

where $D = C/\langle\sigma_{|C}\rangle$ is the quotient curve and R is the ramification divisor. Since deg $(\sigma_{|C}) = 4$ (where σ acts on C as an automorphism of order four) we get:

$$2g(C) - 2 = 4(2g(D) - 2) + \deg R,$$

thus

$$\frac{2g(C) + 6 - \deg R}{8} = g(D).$$

Such that deg $R = 3(N^{'}) + 2h$ (since σ acts as an order four automorphisms on C) where $N^{'}$ denotes the number of fixed points by σ on C and $2h$ the number of interchanged points by σ on it. Since $g(D) \in \mathbb{Z}_{\geq 0}$ we have:

$$2g(C) + 6 - 3N^{'} - 2h \equiv 0 \mod 8. \tag{I}$$

We will now discuss each case of the previous table separately to obtain the complete classification of order 8 automorphisms σ when $m = 0$. At first, it is useful to recall some of the relations which will be used later, that are: $k_{\sigma^2} = k + n_4/2 + 2a$, $N^{'}_{\sigma^2} = N^{'} + 2h$ and $N_1 = (n_2 + n_3) - N^{'} + 4A$ where $2A$ is the number of smooth rational curves which are exchanged by σ and invariants with two isolated fixed points of σ^2 (see Remark 2.1.9).

$\underline{g(C) = 10:}$ Observe that $k_{\sigma^2} = 0$ in this case, thus by Proposition 2.1.8 we have $n_2 + n_3 = 2$ where $k = 0$. Since $N^{'}_{\sigma^2} = 2$ by (I) we have that $(N^{'}, h) = (0, 1)$ (where $N^{'}_{\sigma^2} = N^{'} + 2h$). Hence the two fixed points by σ are contained in the smooth rational curve fixed by σ^4 and invariant by σ^i; $i = 1, 2$. So that $n_2 = n_3 = 1$ by Remark 2.1.11, and $n_4 = 2$ by Proposition 2.1.8. This gives a contradiction with $k_{\sigma^2} = 0$. In fact the two fixed points of type $P^{4,5}$ are contained in a fixed rational curve of σ^2 (see Remark 2.1.9).
$\underline{g(C) = 9:}$ Since $N^{'}_{\sigma^2} = 4$ by (I) we get that $(N^{'}, h) = (2, 1)$. On other hand, by Proposition 2.1.8 and since $k_{\sigma^2} = k = 0$ we have $(n_2, n_3, n_4) = (2, 0, 0)$. So that the two fixed points by σ on C are of type $P^{2,7}$.
$\underline{g(C) = 7:}$ Since $N^{'}_{\sigma^2} = 4$ by (I) we have that $(N^{'}, h) = (4, 0), (0, 2)$ (where $N^{'}_{\sigma^2} = N^{'} + 2h$). Observe that if $(N^{'}, h) = (4, 0)$ and $k = 0$, then $n_4 = 2$ since $k_{\sigma^2} = 1$. By Proposition 2.1.8 we get $n_2 = n_3 = 1$ which contradicts $N^{'} = 4$ (in fact $N^{'} \leq (n_2 + n_3)$ since the fixed points by σ on C are of type $P^{2,7}$ and $P^{3,6}$). If $k = 1$ then $n_4 = 0$ (where $k_{\sigma^2} = k = 1$) and so $(n_2, n_3) = (5, 1)$ by Proposition 2.1.8. Observe that $r = 6 = r_{\sigma^2}$ and $l = 0$ by Proposition 2.1.8, hence σ acts trivially on Pic(X) in this case. On the other hand, if $(N^{'}, h) = (0, 2)$ then $n_2 + n_3 = 2 = N_1$. Thus $n_2 = n_3 = 1$ since this two fixed points are contained in smooth rational curve (see Remark 2.1.11). Thus $k = 0$ and $n_4 = 2$ by Proposition 2.1.8. In this case we have that Pic(X) $=$ S$(\sigma^2) \cong$ U $\oplus$ D$_4$. Thus by Proposition 2.2.1 and since $g(C) > 4$ we know that the $K3$ surface X carries a $\sigma-$invariant elliptic fibration with singular fiber of type I_0^*,such that C intersects each fiber at three points, thus it meets three components of I_0^* of multiplicity one. Since $N^{'} = 0$ and $h = 2$ we get that the four fixed points of σ^2 on C are exchanged two by two by σ, but this is not possible since the fibration is $\sigma-$invariant.
$\underline{g(C) = 6, k_{\sigma^2} = 1:}$ Using the same argument we get that $(N^{'}, h) =$ $(2, 2)$, $(6, 0)$ and $(n_2, n_3, n_4, k) =$ (1,1,2,0), (5,1,0,1) respectively, where $N^{'}_{\sigma^2} = 6$ in this case. Observe that $l = 0$ when $(n_2, n_3, n_4) = (5, 1, 0)$ by Proposition 2.1.8. Thus σ acts trivially on Pic(X) in this case.

$\underline{g(C) = 6, k_{\sigma^2} = 2:}$ Since $N'_{\sigma^2} = 2$ by (I) we get $(N', h) = (0, 1)$. This means that σ does not fix any point on C and the fixed points by it are contained in smooth rational curves. Thus $n_2 = n_3$ by Remark 2.1.11, so that $n_4 = 2+2k$ by Proposition 2.1.8 which contradicts the equality $k_{\sigma^2} = 2 = k + n_4/2 + 2a$. In fact we get $2 = 2k + 1$ by the previous relations.
$\underline{g(C) = 5:}$ Since $N'_{\sigma^2} = 4$ by (I) we get $(N', h) = (2, 1)$. Since $k_{\sigma^2} = 2$ by the relation $k_{\sigma^2} = k + n_4/2 + 2a$ and Proposition 2.1.8 we have the following possibilities: $(k, a, n_2, n_3, n_4) = (1, 0, 4, 2, 2), (0, 0, 0, 2, 4)$ and $(0, 1, 2, 0, 0)$ where the case $k = 2$ is not possible since otherwise $n_2 + n_3 = 10 > N_{\sigma^2}$.
$\underline{g(C) = 4:}$ Since $N'_{\sigma^2} = 6$ thus either $(N', h) = (0, 3)$ or $(4, 1)$. Using the same argument we obtain that the case $(N', h) = (0, 3)$ is not possible. Observe that since $N' = 4$ we get $n_2 + n_3 \geq 4$ thus $k \geq 1$ by Proposition 2.1.8. On the other hand, $k \neq 2$ since otherwise $n_2 + n_3 = 10 > N_{\sigma^2}$ by Proposition 2.1.8 again. Thus $k = 1$ and $(n_2, n_3, n_4) = (4, 2, 2)$ where $k_{\sigma^2} = 2$.
$\underline{g(C) = 3, k_{\sigma^2} = 2:}$ Since $N'_{\sigma^2} = 8$ by (I) we get $(N', h) = (2, 3)$. Observe that $n_2 + n_3 = N' = 2$ where $N'_{\sigma^2} = N_{\sigma^2} = 8$ in this case. So that $k = 0$ by Proposition 2.1.8 and either (k, a, n_2, n_3, n_4) equals (0,1,2,0,0) or (0,0,0,2,4).
$\underline{g(C) = 3, k_{\sigma^2} = 3:}$ Since $N'_{\sigma^2} = 4$ by (I) we have that $(N', h) = (0, 2), (4, 0)$. On the other hand, since $k_{\sigma^2} = 3$ we get $k \leq 2$. In fact k cannot be equal 3 since otherwise $n_2+n_3 = 14 > N_{\sigma^2}$ by Proposition 2.1.8, which is impossible. If $k = 2$ then $(n_2, n_3) = (7, 3)$ by Proposition 2.1.8 where $n_4 = 2$ in this case (since $k_{\sigma^2} = 3 = n_4/2 + k + 2a$). Thus $N'_{\sigma^2} = N' = 4$ since $n_2 + n_3 = 10 = N_{\sigma^2}$. Observe that $r = r_{\sigma^4} = 14, l = 0$ by Proposition 2.1.8 and so σ acts trivially on Pic(X).
If $k = 1$ then (n_4, a) either equals $(0, 1)$ or $(4, 0)$. If $n_4 = 0$ by Proposition 2.1.8 we get $(n_2, n_3) = (5, 1)$ so we have $(N', h) = (4, 0)$. If $n_4 = 4$ then $(n_2, n_3) = (3, 3)$ by Proposition 2.1.8 and that corresponds to $(N', h) = (0, 2)$. Both cases $(N', h) = (4, 0) = (0, 2)$ are possible.
Finally if $k = 0$ then $n_2 + n_3 = 2$ by Proposition 2.1.8. Thus $(N', h) = (0, 2)$ (where the fixed points by σ on C are of type $P^{2,7}$ or $P^{3,6}$). So that $n_2 = n_3 = 1$ by Remark 2.1.11 (since the two fixed points are contained in a smooth rational curve). Thus $n_4 = 2$ by Proposition 2.1.8 and we are in the case $(k, n_4, a) = (0, 2, 1)$.
$\underline{g(C) = 2, k_{\sigma^2} = 3:}$ Since $N'_{\sigma^2} = 6$ we have $(N', h) = (2, 2)$. As we have seen before, the case $k = 3$ is not possible since $n_2 + n_3 > N_{\sigma^2}$, and if $k = 2$ by Proposition 2.1.8 we get $n_2 + n_3 = 10 = N_{\sigma^2}$ which is also not possible since $N'_{\sigma^2} \neq N'$. If $k = 1$ then $n_2 + n_3 = 6$ by Proposition 2.1.8. Since $N' = 2$ the last four fixed points by σ (of type $P^{2,7}, P^{3,6}$) are contained in smooth rational curves, so that $n_2, n_3 \geq 2$ (see Remark 2.1.11). Hence $(n_2, n_3, n_4) = (3, 3, 4)$ by Proposition 2.1.8 and we are in the case $(k, n_4, a) = (1, 4, 0)$. If $k = 0$ then $n_2 + n_3 = 2 = N'$. Observe that $n_4 \leq 4$ by Proposition 2.1.8 and so $(k, a, n_2, n_3, n_4) = (0, 1, 1, 1, 2)$.
$\underline{g(C) = 2, k_{\sigma^2} = 4:}$ Since $N'_{\sigma^2} = 2$ by (I) we get $(N', h) = (0, 1)$, so that the fixed points by σ of type $P^{2,7}, P^{3,6}$ are contained in smooth rational curves. Thus $n_2 = n_3 = 1 + 2k$ and $n_4 = 2 + 2k$ by Proposition 2.1.8 (see Remark 2.1.11). On other hand, we have that $k_{\sigma^2} = k + n_4/2 + 2a$ thus by previous remark we get $4 = 2(k + a) + 1$ which gives a contradiction. Hence the case $g(C) = 2, k_{\sigma^2} = 4$ is not possible.
$\underline{g(C) = 1, k_{\sigma^2} = 4:}$ Since $N'_{\sigma^2} = 4$ by (I) we get $(N', h) = (2, 1)$. Since σ^2 fixes an elliptic curve C the $K3$ surface X carries an elliptic fibration $\pi_C : X \longrightarrow \mathbb{P}^1$ has C as a smooth fiber (as we have seen in Section 2.3). Moreover, the fibration π_C is invariant by

σ^i; $i = 1, 2, 4$ and σ acts with order 8 on the basis of the fibration and fixes two points on it corresponding to the elliptic curve C and another fiber C'. All curves fixed by σ^i are contained in the fibers of π_C, that because they are disjoint with C. Since $k_{\sigma^2} = 4$ the fiber C' is reducible and it has four fixed rational curves by σ^2. So by checking all the possibilities of Kodaira types for fibers one gets that C' is of type I_M such that $M \geq 8$. On the other hand, σ^2 preserved each component in C' since $a_{\sigma^2} = 0$. So that by apply Lemma 2.1.10 we get that C' is of type I_{16}. If $k \neq 0$, then σ would preserve each components of C'. Thus $a = 0$, $k = 2, (n_2, n_3, n_4) = (6, 4, 4)$ by Lemma 2.1.10. Otherwise $k = 0$ thus either $n_4 = 4, a = 1$ so σ acts as a reflection on C', or $n_4 = 0, a = 2$ then σ acts on C' as a rotation with no fixed point on it. This cases appeared in Table 2.3.

Finally, to find the invariants r, l and m_1 of σ we use the facts that $r + l = r_{\sigma^2}, 2m_1 = m_{\sigma^2}$ and the relation $N = 2 + r - l - 2k$ in Proposition 2.1.8. □

Corollary 2.4.2. *Let σ be a purely non-symplectic automorphism on a $K3$ surface X such that σ acts trivially on $Pic(X)$ (i.e $l = m = 0$). Then $k > 0$ and $a = A = h = 0$, moreover all cases in the table do exist.*

m_1	r	N'	N	(n_2, n_3, n_4)	g_{σ^4}	k	$S(\sigma^4)$
4	6	4	6	(5,1,0)	7	1	$U \oplus D_4$
4	6	6	6	(5,1,0)	6	1	$U(2) \oplus D_4$
2	14	4	12	(7,3,2)	3	2	$U \oplus D_4 \oplus E_8$

Table 2.5: The case rk Pic(X) = S(σ) = r .

Proof. These cases are obtained by Theorem 2.4.1. On other hand, we can find these cases directly from the assumption $l = 0$ without using Theorem 6.1 in [2], see Appendix B where we have given an independent proof (not based on the classification of order four automorphisms [2]) of propositions in [14, §5].
On the other hand, we give examples showing the existence of all these cases in Section 2.8.

□

2.5 The curve C is of genus $g > 1$.

We now assume that the curve $C \subseteq \mathrm{Fix}(\sigma^4)$ is of genus $g > 1$ (the case when $g = 1$ has already been studied in § 3). Thus the other curves fixed by σ^4 are smooth rational by Theorem 2.1.4. If the curve C is contained in the fixed locus $\mathrm{Fix}(\sigma^i)$ for $i \in \{1, 2, 4\}$ then we denote by g_{σ^i} its genus (g_σ stands for g_{σ^1}).

2.5.1 The curve C is contained in $\mathrm{Fix}(\sigma^2)$.

In this part we prove first that the genus of curves in $\mathrm{Fix}(\sigma)$ is at most one, then we classify the case with $\mathrm{Fix}(\sigma^2)$ contains a fixed curve C of genus $g(C) > 1$. We recall some notation that will be used here. Let N' denotes the number of fixed points are contained in C, $4s = 2a_{\sigma^2}$ denotes the number of smooth curves that are permuted by σ, interchanged by σ^2 and fixed by σ^4, and let N_{σ^2} be the number of isolated fixed points by σ^2.

Theorem 2.5.1. *Let X be a $K3$ surface and σ be a purely non-symplectic automorphism of order eight on it such that* $\mathrm{Pic(X)} = \mathrm{S}(\sigma^4)$. *Then if $C \subset \mathrm{Fix}(\sigma)$ we get $g(C) \leq 1$. Moreover we have that if σ^2 fixes a curve of genus $g_{\sigma^2} > 1$ then the invariants associated to σ are given in Table 2.6 .*

m_1	m	r	l	(n_2, n_3, n_4)	N	N'	k	a	s	g_{σ^2}	$S(\sigma^4)$
3	3	3	1	$(1,1,2)$	4	2	0	0	0	2	$U \oplus A_1^{\oplus 8}$
2	4	4	2	$(1,1,2)$	4	2	0	0	1	2	$U \oplus A_1^{\oplus 4} \oplus E_8$
											$U \oplus D_4 \oplus D_8$

Table 2.6: The case $g_{\sigma^2} > 1$.

Proof. If σ fixes a curve C of genus $g_\sigma > 1$, then the curve C is also contained in $\mathrm{Fix}(\sigma^2)$ (since $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\sigma^2)$) so that by [2, Theorem 4.1] we get $k_{\sigma^2} = 0$. Thus $k = 0$ and

$$\alpha = \sum_{C_i \subset \mathrm{Fix}(\sigma)} (1 - g(C_i)) = (1 - g(C)) \leq -1.$$

This gives the inquality $(n_2 + n_3) \leq -2$ by Proposition 2.1.8 which is clearly not possible. Hence if C is contained in $\mathrm{Fix}(\sigma)$ then $g(C) = 0, 1$.

We now assume that $\mathrm{Fix}(\sigma^2)$ contains a curve C of genus $g_{\sigma^2} > 1$, then the invariants of σ^2 are given in Table 2 of [2]. Since $m_{\sigma^2} = 2m_1$ is an even number and $a_{\sigma^2} \in 2\mathbb{Z}$ by Remark 2.1.7, so there are just two possible cases of invariants associated to σ^2 that are $(g_{\sigma^2}, a_{\sigma^2}, r_{\sigma^2}, N_{\sigma^2}) = (2, 0, 4, 2)$ and $(2, 2, 6, 2)$. Since $g_{\sigma^2}(C) = 2$ in these two cases, then by Riemann-Hurwitz formula we have that:

$$2g(C) - 2 = \deg(\sigma_{|C})(2g(D) - 2) + \mathrm{degR},$$

where $D = C/\langle \sigma_{|C} \rangle$ and R is the divisor of ramification. Since $\deg(\sigma_{|C}) = 2$ (where σ acts as an involution on C) we get:

$$2g(C) - 2 = 2(2g(D) - 2) + \mathrm{degR},$$

and so

$$\frac{6 - \mathrm{degR}}{4} = g(D),$$

such that $\mathrm{degR} = \mathrm{N}'$ in this case, since σ acts as an involution on C. Hence σ has either 2 or 6 fixed points of type $P^{4,5}$ on it (i.e $N' = n_4 = 2$ or 6 where $k_{\sigma^2} = 0$). Observe that since $k = 0$ by Proposition 2.1.8 we get $n_4 + 2n_2 = 4$, hence $n_4 = 2 = N'$ and $n_2 = n_3 = 1$.

To find the invariants of the lattice $S(\sigma)$ for the first case using the fact that $(r_{\sigma^2}, l_{\sigma^2}, m_{\sigma^2}) = (4, 6, 6)$ by [2, Theorem 4.1] we get immediately that $2m = l_{\sigma^2} = 6$ and $2m_1 = m_{\sigma^2} = 6$. Moreover we have that $r + l = r_{\sigma^2} = 4$ and $N = 4 = 2 + r - l$ by Proposition 2.1.8, so $r = 3$ and $l = 1$. Using the same argument we obtain the values of r, l, m and m_1 for the second case appearing in Table 2.6. Here $(r_{\sigma^2}, l_{\sigma^2}, m_{\sigma^2}) = (6, 8, 4)$. □

Corollary 2.5.2. *Let X be a $K3$ surface and ρ be a purely non-symplectic automorphism of order 16 acts on it and such that* $\mathrm{Pic(X)} = \mathrm{S}(\rho^8)$. *Then ρ does not fix any curve of genus bigger than one.*

Remark 2.5.3.
- *We can say more generally that the fixed locus of a purely non-symplectic automorphism of order 2^q ; $q \geq 3$ does not contain a curve C of genus $g(C) > 1$.*
- *By Theorem 2.5.1 we get that* $\mathrm{Pic(X)} \cong \mathrm{S}(\sigma^4) = \mathrm{U} \oplus \mathrm{L}$ *such that L isomorphic to a direct sum of root lattices of type A_1, E_8, D_4, D_8. Thus by Proposition 2.2.1 the $K3$ surface X carries a σ^4-invariant elliptic fibration π with unique section fixed by σ^4. On the other hand, by Proposition 2.2.1 again the automorphism $\sigma^i; i = 1, 2$ does not preserve this elliptic fibration since $C \subset$* $\mathrm{Fix}(\sigma^2)$. *Hence for each generic fiber F of π we get F and $\sigma^2(F)$ intersect in 3 points only with multiplicity (1,1,2) such that these three points are also the intersection points with the curve C. This explain why we can not use Proposition 2.2.1 here to prove that $k_{\sigma^2} \geq 1$.*

Remark 2.5.4. *In the Appendix A we see that if f_4 is an homogeneous polynomial of degree 4, that defines a smooth generic surface X in $\mathbb{P}^3$, then it does not exist σ of order 8 acting on X such that f_4 is an affine invariant of σ.*

2.5.2 The curve C is fixed only by σ^4 and $\mathrm{Fix}(\sigma)$ contains a rational curve.

Now let σ be a purely non-symplectic automorphism of order eight on a $K3$ surface X. Such that $2m = l_{\sigma^2} > 0$ and $\mathrm{Fix}(\sigma)$ contains at least one smooth rational curve (i.e. $k > 0$) and the fixed curves by σ^2 are also rational (the case of $\mathrm{Fix}(\sigma)$ contains only isolated points is studied in Section 2.7). Then the fixed locus of σ^2 and of σ^4 are given as follows :

$$\mathrm{Fix}(\sigma^2) = E_1 \cup \cdots \cup E_{k_{\sigma^2}} \cup \{p_1, \ldots p_{N_{\sigma^2}}\},$$

$$\mathrm{Fix}(\sigma^4) = C \cup \{E_1 \cup \cdots \cup E_{k_{\sigma^2}}\} \cup \{G_1, \ldots G_{N_1/2}\} \cup \{F_1 \cup F_1^{'} \cup \cdots \cup F_{a_{\sigma^2}} \cup F_{a_{\sigma^2}}^{'}\}.$$

Where C is a curve of genus $g_{\sigma^4} > 0$, E_i, G_i, F_i are smooth rational curves such that $\sigma^2(F_i) = F_i^{'}$, $\sigma^2(G_i) = G_i$ and each G_i contains exactly two isolated fixed points by σ^2 where N_1 denotes the number of isolated fixed points by σ^2 on smooth rational curves.

We recall the notations that will be used here: we denote by $2h$ the number of interchanged points by σ on the curve C; $2A$ be the number of smooth rational curves which are interchanged by σ and invariants by σ^2 with two isolated fixed points on each one of them. Finally as in the previous section we denote by $N^{'}, N_{\sigma^2}^{'}$ the number of isolated fixed points by σ respectively σ^2 on C.

Remark 2.5.5. *By looking in the fixed locus of the purely non-symplectic automorphism of order eight σ and its square σ^2 and using the fact in Remark 2.1.9 one can get that there are relations between the isolated points and smooth rational curves which are fixed by σ and σ^2 given as follows:*

$$k_{\sigma^2} = k + 2a + n_4/2. \tag{2.5.1}$$

$$N_{\sigma^2} = (n_2 + n_3) + 4A + 2h. \tag{2.5.2}$$

$$N_{\sigma^2}^{'} = N^{'} + 2h. \tag{2.5.3}$$

$$N_1 = (n_2 + n_3) - N^{'} + 4A. \tag{2.5.4}$$

so by Proposition 2.1.8 we have

$$N_1 = 2 + 4(k + A) - N^{'}. \tag{2.5.5}$$

And by computing the difference $\chi(\mathrm{Fix}(\sigma^2)) - \chi(\mathrm{Fix}(\sigma))$ *topologically and using the Lefschetz's formula we get that:*

$$2a + 2A + h = l - m. \tag{2.5.6}$$

Theorem 2.5.6. *Let* σ *be a purely non-symplectic automorphism of order 8 on a* $K3$ *surface* X *such that* $k > 0$ *and the fixed curves by* σ^2 *are rational. Let* $g_{\sigma^4} = g(C) > 1$ *be the genus of the curve* $C \subset \mathrm{Fix}(\sigma^4)$*. Then* σ *fixes exactly one smooth rational curve and we are in one of the cases which appear in Table2.7.*

m	r	l	N	N'	(n_2, n_3, n_4)	a	k	A	h	s	k_{σ^2}	N'_{σ^2}	g_{σ^4}
1	13	3	10	2	(3,3,4)	0	1	0	2	1	3	6	2
1	7	1	6	4	(5,1,0)	0	1	0	0	0	1	4	3
2	8	2	6	4	(5,1,0)	0	1	0	0	0	1	4	3

Table 2.7: The case $g_{\sigma^4} > 1$, $C \not\subseteq \mathrm{Fix}(\sigma^2)$.

Proof. Observe that the automorphism σ acts as an automorphism of order four on C such that the fixed points by σ on C are of type $P^{2,7}$ and $P^{3,6}$. On the other hand, by [2, Theorem 8.1] we get $g_{\sigma^4} \leq m_{\sigma^2}$ and the invariants associated to σ^2 are given in the following table:

$m_{\sigma^2} + l_{\sigma^2}$	k_{σ^2}	$g_{\sigma^4} \leq$	$a_{\sigma^2} \leq$
4	3	3	2
6	2	5	3
8	1	7	4

Observe that if $k_{\sigma^2} = 3$ then $l_{\sigma^2} = m_{\sigma^2} = 2$ and a_{σ^2} is either equals to 0 or 2. In fact we have that $l_{\sigma^2} = 2m > 0$ and $m_{\sigma^2} = 2m_1 > 0$ are even numbers, and $2a_{\sigma^2} = 4s \in 4\mathbb{Z}$ by Remark 2.1.7.

To compute the number of fixed points of σ^2 on C we replace the possible values of the invariants of σ^2 in relations (1) and (2) of [2, Theorem 8.1] that are :

$$g_{\sigma^4} - 2a_{\sigma^2} = m_{\sigma^2} - l_{\sigma^2} + 1 - N'_{\sigma^2}/2,$$

$$4a_{\sigma^2} \leq 8 - 2k_{\sigma^2} + N'_{\sigma^2} + l_{\sigma^2} - m_{\sigma^2}.$$

Where $g_{\sigma^4} \leq m_{\sigma^2}$ in this case as we have seen previously. We show for example that for $(l_{\sigma^2}, m_{\sigma^2}, k_{\sigma^2}, a_{\sigma^2}, g_{\sigma^4}) = (2, 2, 3, 0, 3)$ we get $N'_{\sigma^2} = -4$ which is not possible, and if $(l_{\sigma^2}, m_{\sigma^2}, k_{\sigma^2}, a_{\sigma^2}, g_{\sigma^4}) = (2, 6, 1, 4, 3)$ then $N'_{\sigma^2} = 12$, cotradicting the fact that $N_{\sigma^2} \geq N'_{\sigma^2}$ where $N_{\sigma^2} = 6$ by [2, Proposition 1]. By a similar argument we find the possibilities for the invariants of σ^2 which appears in the table below :

N_{σ^2}	k_{σ^2}	a_{σ^2}	g_{σ^4}	l_{σ^2}	m_{σ^2}	N'_{σ^2}
10	3	2	2	2	2	6
8	2	0	$(3, 2)$	2	4	$(0, 2)$
		2	3	2	4	8
6	1	0	≤ 5	2	6	$(10 - 2g_{\sigma^4})$
		2	3	4	4	4
		4	2	6	2	6

By Riemann-Hurwitz formula applied to the automorphism σ on C we have that :

$$2g(C)-2=\deg(\sigma_{|C})(2g(D)-2)+\deg R,$$

where $D=C/<\sigma_{|C}>$ and R is the divisor of ramifiction. Since σ acts on C as an automorphism of order four we get $\deg(\sigma_{|C})=4$ and $\deg R=3N^{'}+2h$, where we denoted by $N^{'}$ the number of fixed points by σ on C and by $2h$ the number of interchanged points by σ on it. Hence

$$2g(C)+6-(3N^{'}+2h)\equiv 0 \mod 8, \tag{I}$$

since $g(D)\in\mathbb{Z}_{\geq 0}$. Now we give a detailed explanation of each of the cases in the previous table separately.

$(g_{\sigma^4},N^{'}_{\sigma^2},k_{\sigma^2},a_{\sigma^2})=(2,6,3,2)$: By Riemann-Hurwitz formula and since σ^2 fixes 6 points on C where $N^{'}_{\sigma^2}=6$ we find that σ exchanges two by two four points on C and fixes the last two. On the other hand, by (2.5.4) of Remark 2.5.5 we have $N_1=4=(n_2+n_3)+4A-2$, thus $(n_2+n_3)\leq 6$. While by Proposition 2.1.8 we have $(n_2+n_3)\geq 6$ since $k\geq 1$. Hence $(n_2+n_3)=6$ and $k=1$ such that two of these six points are on C and the other four are contained in two rational curves that are fixed by σ^4 and invariants by σ^2. The two smooth rational curves fixed by σ^2 (here $k_{\sigma^2}=3$ and $k=1$) either are invariants by σ with two fixed points of type $P^{4,5}$ on each one of them, i.e $(n_4,a)=(4,0)$, so $(n_2,n_3)=(3,3)$ by Proposition 2.1.8, or they are exchanged by σ so that $(n_4,a)=(0,1)$ and $(n_2,n_3)=(5,1)$ by Proposition 2.1.8 again. Observe that the second case is not possible by Remark 2.1.11 since a smooth rational curve which is invariant by σ and σ^2 contains exactly one fixed point of type $P^{2,7}$ and another one of type $P^{3,6}$.

$(g_{\sigma^4},N^{'}_{\sigma^2},k_{\sigma^2},a_{\sigma^2})=(3,0,2,0)$: Since $N^{'}_{\sigma^2}=0$ we have $(N^{'},h)=(0,0)$ (where $N^{'}_{\sigma^2}=N^{'}+2h$) and that is not possible by Riemann-Hurwitz formula.

$(g_{\sigma^4},N^{'}_{\sigma^2},k_{\sigma^2},a_{\sigma^2})=(2,2,2,0)$: By (I) and since $N^{'}_{\sigma^2}=2$ we get $(N^{'},h)=(0,1)$, so that all the fixed points by σ are contained in smooth rational curves that are fixed by σ^4, thus $n_2=n_3$ by Remark 2.1.11 and so $n_4=2+2k$ by Proposition 2.1.8. On the other hand, since $k>0$ then $n_4\geq 4$. So that $k_{\sigma^2}=k+n_4/2+2a>2$ which contradicts $k_{\sigma^2}=2$.

$(g_{\sigma^4},N^{'}_{\sigma^2},k_{\sigma^2},a_{\sigma^2})=(3,8,2,2)$:By relation (I) and since $N^{'}_{\sigma^2}=8$ we get $(N^{'},h)$ =(2,3). On the other hand, since $k>0$ then $n_2+n_3\geq 6$ by Proposition 2.1.8. So that σ fixes at least four points on two rational curves (where $N^{'}=2$) which is not possible. In fact all the fixed points by σ^2 are contained in C so that $N_1=N_{\sigma^2}-N^{'}_{\sigma^2}=0$ (see relation (2.5.4) in Remark 2.5.5).

$(g_{\sigma^4},N^{'}_{\sigma^2},k_{\sigma^2},a_{\sigma^2})=(5,0,1,0)$: By relation (I) and since $N^{'}_{\sigma^2}=0$ we get $(N^{'},h)$ =(0,0), so that the isolated fixed points by σ are contained in smooth rational curves. This means that $n_2=n_3$ (see Remark 2.1.11), thus by Proposition 2.1.8 we have $n_4=2+2k$ which is clearly not possible since $k_{\sigma^2}=k=1$ (where $k>0$).

$(g_{\sigma^4},N^{'}_{\sigma^2},k_{\sigma^2},a_{\sigma^2})=(4,2,1,0)$: Since $N^{'}_{\sigma^2}=2$ by relation (I) we get $(N^{'},h)=(2,0)$. On the other hand, since $k_{\sigma^2}=k=1$ (where $k>0$) we get $n_4=0$, so that $(n_2,n_3)=(5,1)$ by Proposition 2.1.8 which gives a contradiction. In fact σ fixes four points on two smooth rational curves where $(N,N^{'})=(6,2)$ thus $n_2,n_3\geq 2$ by Remark 2.1.11.

$(g_{\sigma^4},N^{'}_{\sigma^2},k_{\sigma^2},a_{\sigma^2})=(3,4,1,0)$: By relation (I) and since $N^{'}_{\sigma^2}=4$ we get $(N^{'},h)$ either equal to (4,0) or (0,2). Observe that since $k_{\sigma^2}=k=1$ (where $k>0$) we get $n_4=0$ and so $(n_2,n_3)=(5,1)$ by Proposition 2.1.8. Hence the case $(N^{'},h)=(0,2)$ is not possible since the six fixed points by σ are contained in smooth rational curves and so $n_2=n_3$ (see

Remark 2.1.11). The case $(N', h) = (4, 0)$ is possible and appears in Table 2.7.
$\underline{(g_{\sigma^4}, N'_{\sigma^2}, k_{\sigma^2}, a_{\sigma^2}) = (3, 4, 1, 2)}$: By the same argument in the previous case we get that (N', h)=(4,0) and $(n_2, n_3, n_4) = (5, 1, 0)$.
$\underline{(g_{\sigma^4}, N'_{\sigma^2}, k_{\sigma^2}, a_{\sigma^2}) = (2, 6, 1, 0)}$: Since $N'_{\sigma^2} = 6$ by relation (I) we get (N', h) =(2,2), hence σ fixes four points on two smooth rational curves and so $n_2, n_3 \geq 2$ (see Remark 2.1.10). This is not possible since $(n_2, n_3, n_4) = (5, 1, 0)$ by Proposition 2.1.8 where $k_{\sigma^2} = k = 1$. By the same argument we can exclude the case $\underline{(g_{\sigma^4}, N'_{\sigma^2}, k_{\sigma^2}, a_{\sigma^2}) = (2, 6, 1, 4)}$.

Using the facts that $r+l = r_{\sigma^2}$, $2m = l_{\sigma^2}$, $2m_1 = m_{\sigma^2}$ and the relation $N = 2+r-l-2k$ in Proposition 2.1.8 it is easy to compute the invariants r, l, m, m_1 of σ. More precisely, the invariants of σ in this table verify the relation (2.5.6) in Remark 2.5.5.

□

2.6 The fixed locus $\mathrm{Fix}(\sigma^4)$ only contains rational curves.

In this section we assume that $\mathrm{Fix}(\sigma^4)$ contains only smooth rational curves and that at least one of them is fixed by σ (i.e $k > 0$).

Theorem 2.6.1. *Let X be a $K3$ surface and σ be a purely non-symplectic automorphism of order eight on it. If $\mathrm{Fix}(\sigma)$ contains a smooth rational curve and all curves fixed by σ^4 are rational, then $(k, A, N, a) = (1, 1, 10, 0)$ and $(n_2, n_3, n_4) = (3, 3, 4)$. The corresponding invariant of $S(\sigma^4)$ are $(r, l, m) = (13, 3, 1)$.*

Proof. By [2, Theorem 5.1] and by excluding the cases when m_{σ^2} and a_{σ^2} are odd numbers (since $m_{\sigma^2} = 2l$ and $a_{\sigma^2} \in 2\mathbb{Z}$ by Remark 2.1.7) we get that the possible invariants for σ^2 are $(r_{\sigma^2}, m_{\sigma^2}, N_{\sigma^2}, k_{\sigma^2}, a_{\sigma^2}) = (10, 4, 6, 1, 0)$, $(16, 2, 10, 3, 0)$ and $(12, 2, 6, 1, 2)$. Observe that since the involution σ^4 fixes only rational curves, all isolated fixed points by σ are contained in smooth rational curves. So that $n_2 = n_3$ by Remark 2.1.11 (where each invariant rational curve by σ either contains two points of type $P^{4,5}$ or one point of type $P^{2,7}$ and another one of type $P^{3,6}$). Hence by Proposition 2.1.8 we have that $n_2 = 1+2k$ and $n_4 = 2+2k$. Since $k > 0$ then $n_4 \geq 4$ which gives that $k_{\sigma^2} \geq 3$ (where $k_{\sigma^2} = k + n_4/2 + 2a$, see Remark 2.1.9). That excludes the first and the third cases of the possible invariants of σ^2. For the second case we have that $k = 1$, $n_4 = 4$ and $a = 0$ since $k_{\sigma^2} = 3$. So we get $n_2 = n_3 = 3$. Moreover we have that $2m = l_{\sigma^2} = 2, 2m_1 = m_{\sigma^2} = 2, r + l = r_{\sigma^2} = 16$ and by Proposition 2.1.8 we have $r - l = 6$.

□

2.7 The fixed locus $\mathrm{Fix}(\sigma)$ has only isolated points.

Let σ be a purely non-symplectic automorphism of order eight on a $K3$ surface X that fixes only isolated points and $m > 0$ (the case $m = 0$ is discussed in Section 2.4) and the genus of the fixed curves by σ^4 is $g(C) \neq 1$ (the case with $g(C) = 1$ and $k = 0$ is studied in Section 2.3). It follows from Proposition 2.1.8 that $\mathrm{Fix}(\sigma)$ contains exactly two points q_1, q_2 of type $P^{2,7}$ or of type $P^{3,6}$, or one point of each type. Moreover, the fixed locus of σ^2 is either as :

$$\mathrm{Fix}(\sigma^2) = C \cup (F_1 \cup F'_1) \cdots \cup (F_a \cup F'_a) \cup\ E_1 \cup \ldots E_{\frac{n_4 - N'}{2}} \cup \{q_1, q_2, p_1, \ldots, p_{4A}\},$$

if σ^2 fixes a curve C of genus $g(C) > 1$, or as follows:

$$\mathrm{Fix}(\sigma^2) = (F_1 \cup F_1^{'}) \cdots \cup (F_a \cup F_a^{'}) \cup \; E_1 \cup \ldots E_{n_4/2} \cup \{q_1, q_2, p_1, .., p_{4A}\} \cup \{p_1^{'}, .., p_{2h}^{'}\}.$$

Where each E_i is a smooth rational curve containing 2 fixed points by σ of type $P^{4,5}$, $N^{'}$ denotes the number of the remaining fixed points on C , and $\{p_1^{'}, \ldots, p_{2h}^{'}\}$ are the points on C that are exchanged by σ and fixed by σ^2.

Theorem 2.7.1. *Let σ be a purely non-symplectic automorphism of order eight on a $K3$ surface X having only isolated fixed points, such that $m > 0$ and the genus of the fixed curves by σ^4 is $g(C) \neq 1$. Then σ fixes at most 6 points and we are in one of the cases appearing in Table 2.8.*

m	r	l	s	a	A	h	N	$N^{'}$	(n_2, n_3, n_4)	k_{σ^2}	$N^{'}_{\sigma^2}$	g_{σ^4}
1	2	2	0	0	0	1	2	2	(2,0,0)	0	4	5
1	9	7	1	1	1	2	4	2	(1,1,2)	3	6	2
3	7	5	2	0	0	2	4	2	(1,1,2)	1	6	2
1	6	6	1	1	0	3	2	2	(2,0,0)	2	8	3
2	6	4	1	0	0	2	4	0	(1,1,2)	1	4	3
1	8	4	1	0	0	3	6	2	(0,2,4)	2	8	3
1	5	3	0	0	1	0	4	0	(1,1,2)	1	0	5
			0	0	1	0	4	2	(1,1,2)	1	2	4
			0	0	0	2	4	0	(1,1,2)	1	4	3
			0	0	0	2	4	2	(1,1,2)	1	6	2
2	6	4	0	0	1	0	4	0	(1,1,2)	1	-	0
3	7	5	1	0	1	0	4	0	(1,1,2)	1	-	0
1	9	7	0	1	2	0	4	0	(1,1,2)	3	-	0

Table 2.8: The case $k = \alpha = 0, m > 0$.

Proof. By Proposition 2.1.8 since $\alpha = k = 0$ we get that $(n_2 + n_3) = 2$ and $n_4 \leq 4$ where $2n_3 = n_4$ in this case. More precisely the possible cases for (n_2, n_3, n_4) are (2,0,0,2) , (1,1,2,4) and (0,2,4,6). We can find the invariants related to σ by discussing the different cases for the fixed locus of σ^2 and σ^4.

σ^4 fixes a curve C of genus $g(C) > 1$:

We can assume that $C \not\subseteq \mathrm{Fix}(\sigma^2)$ otherwise we have discussed this case already in Theorem 2.5.1. Since C is fixed by σ^4 then C is also invariant by σ. Hence σ acts on C as an automorphism of order four with at most two isolated fixed points on it. In fact the fixed points by σ on C are of type $P^{2,7}, P^{3,6}$ thus $N^{'}$ either equals 0 or 2 (since $(n_2 + n_3) = 2$). On the other hand, by Riemann-Hurwitz formula applied to the automorphism σ on C we have that:

$$2g(C) - 2 = 4(2g(D) - 2) + \deg R,$$

thus

$$\frac{2g(C) + 6 - \deg R}{8} = g(D).$$

Such that deg$R = 3N' + 2h$ (as we have seen in Theorem 2.5.6). Since $g(D) \in \mathbb{Z}_{\geq 0}$ we get:

$$2g(C) + 6 - 3N' - 2h \equiv 0 \mod 8. \tag{I}$$

Here we distinguish the two following cases:

i) Fix(σ^2) has only isolated fixed points :
By relations (2.5.1) and (2.5.2) of Remark 2.5.5 we have that $k_{\sigma^2} = 2a + n_4/2$ and $N_{\sigma^2} = 2 + 4A + 2h$ (since $k = 0$ and so $n_2 + n_3 = 2$ by Proposition 2.1.8). Since $k_{\sigma^2} = 0$ by [2, Proposition 1] we get $N_{\sigma^2} = 4$. This gives $a = n_4 = A = 0$ and $h = 1$. Since $n_4 = 0$ by Proposition 2.1.8 we have $(n_2, n_3) = (2, 0)$. Observe that $N' = 2$, otherwise σ would fix two points of type $P^{2,7}$ on a rational curve and this is not possible by Remark 2.1.11. Hence by (I) we get that the genus of the curve $C \subseteq$ Fix(σ^4) is either $g(C) = 5$ or 9. We want to show that $g = 9$ is not possible. One can deduce it easily by the Tables 6 and 5 of [2], but we give here a self-contained argument. Since $2a_{\sigma^2} \in 4\mathbb{Z}$ by Remark 2.1.7 and $k_{\sigma^2} = 0$ we get $k_{\sigma^4} \in 4\mathbb{Z}$. In fact all isolated fixed points by σ^2 are contained in C since $(N', h) = (2, 1)$ and $N_{\sigma^2} = 4$ (here $N_{\sigma^2} = 4 = N' + 2h$) so that $k_{\sigma^4} = 2a_{\sigma^2} \in 4\mathbb{Z}$. Thus by [9, §4], see also [4, Figure 1], we have that $(g_{\sigma^4}, k_{\sigma^4}, r_{\sigma^4}) = (5, 0, 6), (5, 4, 10), (9, 0, 2)$ such that $S(\sigma^4) = U(2) \oplus A_1^{\oplus 4}, U \oplus D_4^{\oplus 2}, U(2)$ respectively. Observe that for the second case by applying Proposition 2.2.1 we know that the $K3$ surface X carries a $\sigma-$invariant jacobian elliptic fibration π with unique section fixed by σ^4, which contradicts the fact that $N' = N = 2$ (i.e. σ does not preserve any rational curve so that the unique section of π is not invariant by σ). Hence we have two possible cases that are $(g_{\sigma^4}, k_{\sigma^4}, r_{\sigma^4}) = (5, 0, 6)$ (this is the first case in the table) and $(g_{\sigma^4}, k_{\sigma^4}, r_{\sigma^4}) = (9, 0, 2)$. Moreover by (2.5.6) of Remark 2.5.5 we get $l = m+1$ (where $A = a = 0$ and $h = 1$) such that $r = l$ by Proposition 2.1.8. Thus $r_{\sigma^4} = r + l + 2m = 4m + 2$. So that $m = 1$, $r = l = 2$ for the first case and $m = 0$, $r = l = 1$ for the second one, observe that the second case is not possible since $m = 0$ (which contradict with our assumption that $m > 0$) and it has already appeared in Table 2.4 of Theorem 2.4.1.

ii) Fix(σ^2) contains smooth rational curve : Since $k_{\sigma^2} > 0, m > 0$ then as we have seen in Theorem 2.5.6 the possible cases of invariants of σ^2 are given in the following table (where N'_{σ^2} denotes here the number of isolated fixed points of σ^2 contained in C) :

N_{σ^2}	k_{σ^2}	a_{σ^2}	g_{σ^4}	l_{σ^2}	m_{σ^2}	N'_{σ^2}
10	3	2	2	2	2	6
8	2	0	(3, 2)	2	4	(0, 2)
		2	3	2	4	8
6	1	0	≤ 5	2	6	$(10 - 2g_{\sigma^4})$
		2	3	4	4	4
		4	2	6	2	6

We have seen, at the beginning of this proof, that $n_4 \leq 4$ and $n_2 + n_3 = 2$, so we get $N' \in \{0, 2\}$ where the fixed points by σ on the curve C are of type $P^{2,7}, P^{3,6}$. Observe that if $N' = 0$ then $n_2 = n_3 = 1$ by Remark 2.1.11 (in fact the fixed points by σ are contained in smooth rational curve in this case) then $n_4 = 2$ by Proposition 2.1.8.

If $\boxed{g(C) = g_{\sigma^4} = 2}$ then by (I) we have that $(N', h) \in \{(0, 1), (0, 5), (2, 2)\}$ and by the previous table we get $(N'_{\sigma^2}, k_{\sigma^2}) \in \{(6, 3), (6, 1), (2, 2)\}$ (recall that $N'_{\sigma^2} = N' + 2h$). If $(N_{\sigma^2}, k_{\sigma^2}) = (2, 2)$ then $(N', h) = (0, 1)$ and we show that this case is not possible. In fact since $N' = 0$ in this case, we get that $(n_2, n_3, n_4) = (1, 1, 2)$ by the previous remark, which is clearly impossible since $k_{\sigma^2} = 2 = 2a + 1$ (recall $k_{\sigma^2} = 2a + k + n_4/2$, $n_4 = 2$ and $k = 0$). On the other hand, if $(N'_{\sigma^2}, k_{\sigma^2})$=(6,3) then $(n_4, a) = (2, 1)$ since $n_4 \leq 4$. Thus $(n_2, n_3) = (1, 1)$ by Proposition 2.1.8. In the same way we get that $(n_2, n_3, n_4) = (1, 1, 2)$ when $(N'_{\sigma^2}, k_{\sigma^2})$=(6,1). These case appear in Table 2.8.

If $\boxed{g(C) = 3}$ then by (I) we have that (N', h)=(0,2),(0,6) or (2,3). By the previous table we get $(N'_{\sigma^2}, k_{\sigma^2})$=(0,2),(8,2),(4,1), so that we have in fact $(N', h) = (0, 2), (2, 3)$ (the case $(N', h) = (0, 6)$ is not possible since it would give $N'_{\sigma^2} = 12$) and the case $(N'_{\sigma^2}, k_{\sigma^2})$=(0,2) is not possible by Riemann-Hurwitz formula. If $(N'_{\sigma^2}, k_{\sigma^2})$= (4,1) then $(n_2, n_3, n_4) = (1, 1, 2)$ by Proposition 2.1.8. Finally if $(N'_{\sigma^2}, k_{\sigma^2})$=(8,2), then either (n_4, a) equals (0,1) then (n_2, n_3)=(1,1) by Proposition 2.1.8, or equals (4,0) then (n_2, n_3) = (0,2) by Proposition 2.1.8 again. On the other hand if $\boxed{g(C) = 4}$ then $(N'_{\sigma^2}, k_{\sigma^2})$=(2,1) by the previous table, so that (N', h) =(2,0) by (I) and $(n_2, n_3, n_4) = (1, 1, 2)$ by Proposition 2.1.8. Finally if $\boxed{g(C) = 5}$ then $(N'_{\sigma^2}, k_{\sigma^2})$ =(0,1), so that $N' = h = 0$ and (n_2, n_3)=(1,1,2) by Proposition 2.1.8.

<u>σ^4 **fixes only rational curves:**</u>

Observe that $k_{\sigma^2} > 0$ where otherwise we have studied this case already in part (i). Thus we are in one of the cases given in [2, Theorem 5.1]. Since $m_{\sigma^2} = 2l$ is an even number and $a_{\sigma^2} \in 2\mathbb{Z}$ by Remark 2.1.7, we have that $(r_{\sigma^2}, m_{\sigma^2}, N_{\sigma^2}, k_{\sigma^2}, a_{\sigma^2}) = (10, 4, 6, 1, 0)$, $(16, 2, 10, 3, 0)$ and $(12, 2, 6, 1, 2)$. Observe that since the fixed points of σ are contained in smooth rational curves by Remark 2.1.11 we get $n_2 = n_3 = 1$, so that $n_4 = 2$ by Proposition 2.1.8. Hence $a = 1$ when $k_{\sigma^2} = 3$ and $a = 0$ otherwise.

Finally, to find the invariants r, l, m and m_1 of σ we use the fact that $r + l = r_{\sigma^2}$, $2m = l_{\sigma^2}$, $2m_1 = m_{\sigma^2}$ and the relation $N = 2 + r - l - 2k$ in Proposition 2.1.8.

□

2.8 Examples:

In this section we give several examples corresponding to several cases in the classification of the non-symplectic automorphisms of order eight. We construct all this examples by using elliptic fibrations on $K3$ surfaces. The Section 1.3 contains the main definitions and properties of elliptic fibrations that we need.

Example 2.8.1. Consider the elliptic fibration $\pi_C : X \longrightarrow \mathbb{P}^1$ with a Weierstrass equation:

$$y^2 = x^3 + a(t)x + b(t).$$

where $a(t) = at^8 + b$ and $b(t) = ct^8 + d$ with $a, b, c, d \in \mathbb{C}$. The fibration π_C admits the order eight automorphism:

$$\sigma(x, y, t) = (x, y, \zeta_8 t).$$

The fibers preserved by σ are over 0 and ∞ and the action of σ at infinity is:

$$(x/t^4, y/t^6, 1/t) \longmapsto (-x/t^4, iy/t^6, \zeta_8^7/t)$$

The discriminant polynomial of π_C is:

$$\Delta(t) := 4a(t)^3 + 27b(t)^2 = h_1 t^{24} + h_2 t^{16} + h_3 t^8 + O(t^4),$$

where

$$h_1 = 4a^3, \quad h_2 = 12a^2b + 27c^2, \quad h_3 = 12ab^2 + 54cd.$$

Observe that $\Delta(t)$ has 24 simple zeros for a generic choice of the coefficients. By studying the zeros of $\Delta(t)$ and looking in the classification of singular fibers of elliptic fibrations on surfaces (see e.g [17, section 3]) one obtains the following: for a generic choice of the coefficients of $a(t)$ and $b(t)$ the fibration has 24 fibers of type I_1 over the zeros of $\Delta(t)$, both fibers over $t = 0$ and $t = \infty$ are smooth elliptic curves, moreover the automorphism σ fixes pointwisely the fiber over 0 and acts as an *order 4 automorphism* on the fiber over ∞. This is the **first case in Table 2.1**. On the other hand, if $h_1 = 0$ ($a = 0$) the fibration acquires a fiber of type IV^* at ∞ by a generic choice of the parameters. This gives an example for the **second case in Table 2.1**.

In this case by using standard transformations on the parameters in the Weierstrass form we get that the number of moduli is 2. In fact both the polynomials $a(t), b(t)$ are depend on 2 parameters, but we can apply the transformation $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$; $\lambda \in \mathbb{C}^*$. to eliminate one of the 4 parameters. Moreover the automorphisms of $\mathbb{P}^1$ commuting with $t \mapsto \zeta_8 t$ are of the form $t \mapsto \mu t$; $\mu \in \mathbb{C}^*$. So we can eliminate a second parameter. This shows that the family depends on 2 parameters. So that generically $\mathrm{rkT_X} = 12$ (where $\mathrm{rkT_X} = 4\mathrm{m_1}$ and m_1 equals to the number of moduli $+1$) and rk Pic(X) = 10. And we remark that by a generic choice of the coefficients the action of σ^4 is trivial on $\mathrm{Pic}(X)$. By Shioda-Tate formula rk Pic(X) $= 2 +$ rk MW $+ \sum_{\mathrm{F:fiber}}(\#\text{components of F} - 1)$ so that the rank of the Mordell-Weil group of π is rk MW $= 8$. This means that the fibration admits sections of infinite order.

Example 2.8.2. Consider the elliptic fibration $\pi_C : X \longrightarrow \mathbb{P}^1$ in Weierstrass form given by :

$$y^2 = x^3 + a(t)x + b(t).$$

where $a(t) = at^8 + b$ and $b(t) = ct^8 + d$ with $a, b, c, d \in \mathbb{C}$. This elliptic fibration carries the non-symplectic automorphism σ of order eight:

$$(x, y, t) \mapsto (x, -y, \zeta_8 t).$$

The fibers preserved by σ are over $0, \infty$ and the action at infinity is

$$(x/t^4, y/t^6, 1/t) \longmapsto (-x/t^4, -iy/t^6, \zeta_8^7/t)$$

The discriminant polynomial of π_C is:

$$\Delta(t) := 4a(t)^3 + 27b(t)^2 = h_1 t^{24} + h_2 t^{16} + O(t^8),$$

where

$$h_1 = 4a^3 \quad \text{and} \quad \mathrm{h_2 = 12a^2b + 27c^2}.$$

For a generic choice of the coefficients of $a(t)$ and $b(t)$ the fibration has 24 fibers of type I_1 over the zeros of $\Delta(t)$ (see [17, section 3]), σ acts as an *involution* on the fiber over 0 and it acts as an *order 4 automorphism* on the fiber over ∞ (both fibers are smooth). So we have an example for the **second case in Table 2.2**. If $h_1 = 0$ $(a = 0)$ the fibration acquires a fiber of type IV^* at ∞ by a generic choice of the parameters. This is the **last case in Table 2.2**. Observe that this fibration is the same as the fibration in Example 2.8.1, considered here with a different automorphism.

Example 2.8.3. Consider the elliptic fibration $\pi_C : X \longrightarrow \mathbb{P}^1$ in Weierstrass form given by

$$y^2 = x^3 + a(t)x + b(t),$$

where $a(t) = at^8 + b$ and $b(t) = ct^4 + dt^{12}$; $a, b, c, d \in \mathbb{C}$. Observe that it carries the order eight automorphism

$$\sigma : (x, y, t) \longmapsto (-x, iy, \zeta_8^7 t).$$

For generic choice of the coefficients the fiber over $t = 0$ is smooth. The automorphism σ^4 is an involution fixing the smooth elliptic curve over $t = 0$ and some rational curves in the singular fiber over $t = \infty$. On the other hand, σ acts on the elliptic curve over $t = 0$ as an *order 4 automorphism* with 2 isolated fixed points. Moreover it acts as the identity on the fiber over $t = \infty$, where the action at infinity is

$$(x/t^4, y/t^6, 1/t) \longmapsto (x/t^4, y/t^6, \zeta_8/t).$$

The discriminant of π_C is :

$$\Delta(t) = h_1 t^{24} + h_2 t^{16} + h_3 t^8 + 4b^3,$$

where

$$h_1 = 4a^3 + 27d^2 \ , \ h_2 = 12a^2b + 54cd \ , \ h_3 = 12ab^2 + 27c^2.$$

Observe that $\Delta(t)$ has 24 simple zeros for a generic choice of the coefficients. By studying the zeros of $\Delta(t)$ and looking in the classification of singular fibers of elliptic fibrations on surfaces (e.g [17, section 3]) one obtains the following cases: for the generic choice of the coefficients of $a(t)$ and $b(t)$ the fibration has 24 fibers of type I_1. If $h_1 = 0$ the fibration has a fiber I_8 at infinity. If $h_1 = h_2 = 0$ we get a fiber I_{16}. By [5, §3] a holomorphic two form is given by $\omega_X = (dt \wedge dx)/2y$ and so the action of σ on it is $\sigma^*(\omega_X) = \frac{-\zeta_8^7}{i}\omega_X = \zeta_8\omega_X$. We can go further than that and determine exactly the type of the local action of σ at the two fixed points on the elliptic curve C. We look at the elliptic fibration locally around the fiber over $t = 0$. The equation in $\mathbb{P}^2 \times \mathbb{C}$ is given by:

$$F(x, y, z, t) := zy^2 - (x^3 + (at^8 + b)z^2x + (ct^4 + dt^{12})z^3) = 0.$$

Where $(x : y : z)$ are the homogeneous coordinates of $\mathbb{P}^2$. Note that the fiber at $t = 0$ is a smooth elliptic curve. In fact the curve defined by the equation

$$f := \{zy^2 - x^3 - z^2x = 0\}$$

is smooth since the partial derivative $\partial f/\partial x, \partial f/\partial y, \partial f/\partial z$ are equal to zero if and only if $x = y = z = 0$ which is impossible. The two fixed points by σ on the fiber over $t = 0$ are $p := (0 : 1 : 0)$ and $p' := (0 : 0 : 1)$. The second one $p' := (0 : 0 : 1)$ is contained in the chart $z = 1$. Moreover p' belongs to the open set $\partial F/\partial x \neq 0$, indeed $F_x(p') := \frac{\partial F(x,y,1,0)}{\partial x} \neq 0$. By [18, §6.4, p.210] the one-form for the elliptic curve in this open subset is:

$$dy/F_x(p') = dy/(-3x^2 - b).$$

The action of σ here is a multiplication by i : $(\sigma^*(dy/F_x(p')) = i(dy/F_x(p')))$, so that the action on the holomorphic 2-form $dt \wedge (dy/(-3x^2(at^8 + b))$ is the multiplication by ζ_8 as expected. In particular the local action at p' is of type (7,2). Similarly we can do the computation on the open subset in the chart $y = 1$ which contains the fixed point p, and we can find again the same action (7,2). Observe that since σ acts as the identity on the fiber over $t = \infty$, it fixes at least one component of this fiber (of the fiber I_8 or I_{16}). This gives an example for **the cases 2 and 6 in Table 2.3**.

On other hand, the fibration π_C admits also the automorphism $\tau(x, y, t) = (-x, -iy, \zeta_8^3 t)$. This automorphism acts also by multiplication by ζ_8 on the holomorphic 2-form ω_X, thus τ is not a power of σ (i.e they are not equivalent). Moreover the square of τ preserves each components of the fiber at $t = \infty$ and fixes at least one of them, where the action at infinity is:

$$(x/t^4, y/t^6, 1/t) \longmapsto (x/t^4, -y/t^6, \zeta_8^5/t).$$

By a similar computation as above one sees that the local action at the fixed points on the fiber C is of type (3,6), so we have an example for **the cases 3 and 7 in Table 2.3** respectively.

Example 2.8.4. *<u>(Translation):</u>*
We give here an example for the cases $(C', a, N') = (I_0, 0, 0)$ in Table 2.2 and $(I_8, 1, 2)$, $(I_{16}, 2, 2)$ in Table 2.3. Observe that in this three cases the non-symplectic automorphism of order 8 on X acts as a translation of order two on the fiber C' (where the square of this translation is the identity on C' since it fixes the smooth elliptic curve C' in the first case and has at least one fixed component in C' for the remaining two cases).

First, we give a short introduction of some basic facts in the geometry of elliptic curves, then we construct the translation of order two that we need.
We take E an elliptic curve with a 2-torsion point corresponding to $(x, y) = (0, 0)$. Then the equation of E has the Weierstrass form (see [17]):

$$y^2 = x(x^2 + ax + b) \; ; \; a, b \in \mathbb{C}. \qquad (1)$$

Let the point at infinity $\mathcal{O} := (0:1:0) = (x:y:z)$ be the zero element for the group law defined on the set of points of E. We denoted by $P := (0:0:1) = (x:y:z)$ the 2-torsion point. In fact $\mathcal{P}+\mathcal{P} = \mathcal{O}$ see [21, Ch II, §1 (a]. So that the following map:

$$\begin{array}{rcccc} \tau & : & E & \rightarrow & E \\ & & \mathcal{Q} & \mapsto & \mathcal{Q}+\mathcal{P}, \end{array}$$

is a translation of order two on E (in fact $\tau^2(\mathcal{Q}) = \mathcal{Q}+2\mathcal{P} = \mathcal{Q}$ since $\mathcal{P}$ is a point of order two). More precisely, by [21, Ch I, §4] the translation τ is given as follows:

$$\tau : (x,y) \mapsto (y^2/x^2 - a - x, y/x.\tau(x)). \tag{2}$$

If we replace x in the equation of E by $\frac{x^{'}-a}{3}$ and y by $y/27$ and calling $x^{'}$ again x, we get a new equation in Wierstrass form for the elliptic curve E which is:

$$y^2 = x^3 + Ax + B, \tag{3}$$

where $A = 9b - 3a^2$ and $B = 3a^2 - 9ab$. It is a well known fact (see [21, Ch II]) that a $K3$ surface with a 2-torsion section has equation:

$$y^2 = x(x^2 + a(t)x + b(t),$$

where $a(t)$ and $b(t)$ are polynomials of degree 4 and 8 respectively. Or equivalently:

$$y^2 = x^3 + A(t)x + B(t),$$

where

$$A(t) = 9b(t) - 3a(t)^2 \text{ and } \mathrm{B(t) = 3a(t)^2 - 9a(t)b(t)}.$$

Now the map:

$$\tau : (x,y,t) \mapsto (y^2/x^2 - a(t) - x, y/x.\tau(x), t), \tag{4}$$

is an automorphism on X such that it acts as a translation of order two on the generic fiber of π (here we denote $\tau(x)$ the first coordinate of $\tau(x,y,t)$).

Consider now the non-symplectic automorphism of order eight on X:

$$\sigma : (x,y,t) \mapsto (-x, iy, \zeta_8^7 t).$$

This automorphism preserves the jacobian elliptic fibration $\pi : X \longrightarrow \mathbb{P}^1$ defined as follows:

$$y^2 = x(x^2 + a(t)x + b(t)),$$

where $a(t) = \alpha t^4, b(t) = \beta t^8 + \gamma$; $\alpha, \beta, \gamma \in \mathbb{C}$. More precisely, by doing the same transformation as in (3) we get that π is given equivalently by:

$$y^2 = x^3 + A(t)x + B(t),$$

such that $A(t) = (9\beta - 3\alpha^2)t^8 + 9\gamma$ and $B(t) = (2\alpha^3 - 9\alpha\beta)t^{12} - 9\alpha\gamma t^4$. The discriminant of π is:

$$\Delta(t) = K(\beta t^8 + \gamma)^2[(\alpha^2 - 4\beta)t^8 - 4\gamma];\ K \in \mathbb{C} \text{ is a constant.}$$

Consider now the translation τ as follows:

$$\tau : (x, y, t) \mapsto (y^2/x^2 - \alpha t^4 - x, y/x.\tau(x), t).$$

As we have seen, τ is an automorphism of X and it acts as a translation of order two on the generic fibers of π. Moreover, one can get easily that $\tau \circ \sigma = \sigma \circ \tau$, thus the $K3$ surface X has the order eight non-symplectic automorphism $\sigma^{'} := \{\sigma \circ \tau\}$. Observe that the automorphisms σ and $\sigma^{'}$ act with order eight on $\mathbb{P}^1$, and they preserve the two fibers over $t = 0$ and $t = \infty$ and act as an automorphism of order four on the smooth fiber over $t = 0$ given by $f := zy^2 - x^3 - 9\gamma xz^2$. Moreover, σ acts as the identity on the fiber over $t = \infty$, while $\sigma^{'}$ acts as an order two translation on it (where the action of σ at infinity is $(x/t^4, y/t^6, 1/t) \mapsto (x/t^4, y/t^6, \zeta_8 1/t)$ and $\sigma^{'2} = (\sigma \circ \tau)^2 = id$).

Studying the zeros of the discriminant $\Delta(t)$, and looking in the classification of singular fibers of elliptic fibrations on surfaces (e.g [17, §3]) we get the following:

-For generic α, β, γ the fibration π has 8 fibers of type I_2 and 8 fibers of type I_1 over the zeros of $\Delta(t)$, $\sigma^{'}$ acts as an order four automorphism on the fiber over 0 and it acts as an order two translation on the fiber at ∞ (both fibers are smooth). This gives an example for the **first case in Table 2.2** (here we suppose that the fiber over $t = 0$ is $C^{'}$ and the fiber at infinity is C).

-If $\alpha^2 - 4\beta = 0$ with $(\beta \neq 0)$, then $\Delta(t) = K(\beta t^8 + \gamma)^2(-4\beta)$; $K \in \mathbb{C}^*$. So that the fibration acquires a fiber of type I_8 over ∞ (in fact $\Delta(t)$ has a zero of order 8 over $t = \infty$ and $A(t), B(t)$ are non zero). The automorphism $\sigma^{'}$ acts as a translation of order two on the fiber I_8, this means that $\sigma^{'}$ does not have any fixed point on I_8 and $\sigma^{'2}$ fixes at least one component on it (since $\sigma^{'}$ acts as an order two translation on it). This corresponds to the **fourth case in Table 2.3**.

-If $\beta = 0$, $(\alpha \neq 0)$, the discriminant $\Delta(t) = K\gamma^2(\alpha^2 t^2 - 4\gamma)$ vanishes at $t = \infty$ with order 16, and $A(t), B(t)$ are nonzero at ∞. Thus we get a fiber of type I_{16}. We are in the **eighth case of Table 2.3** (where $\sigma^{'}$ acts as an order two translation on the generic fiber).

Example 2.8.5. *The case* $(r, l, m) = (6, 0, 0)$, $\mathrm{Pic(X)} = \mathrm{U} \oplus \mathrm{D}_4$ *:*

Consider the $K3$ surface X with elliptic fibration:

$$X : y^2 = x^3 + A(t)x + B(t),$$

where $A(t) = at^2 + bt^6$ and $B(t) = ct^3 + dt^7 + ft^{11}$ with $a, b, c, d, f \in \mathbb{C}$. The fibration $\pi : X \longrightarrow \mathbb{P}^1$ carries the order eight automorphism:

$$\sigma : (x, y, t) \longmapsto (ix, \zeta_8^3 y, it).$$

Observe that the polynomial $A(t)$ depends on 2 parameters and $B(t)$ depends on 3 parameters, but we can apply the transformation $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$; $\lambda \in \mathbb{C}^*$ to eliminate one of the 5 parameters. Moreover the automorphisms of $\mathbb{P}^1$ commuting with $t \mapsto it$ are of the form $t \mapsto \mu t; \mu \in \mathbb{C}^*$. So we can eliminate a second parameter. This shows that the family depends on 3 parameters and we write $A(t)$ and $B(t)$ as follows:

$$A(t) = t^2 + bt^6 \ , \ B(t) = ct^3 + dt^7 + t^{11}.$$

On other hand, the fibration π has a fiber I_0^* over $t = 0$ and a fiber II (see [17, §3]). As we have seen in Example 2.8.3 the holomorphic two-form is given by $\omega_X = (dt \wedge dx)/2y$, so that the action of σ on it is a multiplication by ζ_8. This is a three dimensional family, so we have $\mathrm{rkT_X} = 4\mathrm{m}_1 = 16$ and in fact the family of $K3$ surfaces in Table 2.4 with $(r,l,m) = (6,0,0)$ depends on 3 parameters. So this is in fact the whole family. Hence for the generic choice of the parameters b, c, d the action is trivial on Pic(X). The curve C intersects in three points the fiber I_0^* (at the three multiplicity one components of I_0^*) and meets the fiber II with multiplicity 3 at the singular point.

Observe that the fiber I_0^* contains the smooth rational fixed curve by σ and four fixed points with local action $P^{2,7}$ (see Remark 2.1.11 where σ preserves each smooth rational curve of X since $\mathrm{Pic(X)} = \mathrm{S}(\sigma)$). On other hand the invariant elliptic cuspidal curve over $t = \infty$ contains two fixed points of type $P^{3,6}, P^{2,7}$ respectively.

Example 2.8.6. *The case* $(r,l,m) = (14,0,0)$, $\mathrm{Pic(X)} = \mathrm{U} \oplus \mathrm{D}_4 \oplus \mathrm{E}_8$ *:*

Consider the elliptic fibration $\pi : X \mapsto \mathbb{P}^1$ as in Example 2.8.5 with $b = f = 0$, this is given by:

$$y^2 = x^3 + at^2x + (ct^3 + dt^7),$$

The fibration carries the automorphism of order eight:

$$\sigma : (x, y, t) \longmapsto (ix, \zeta_8^3 y, it).$$

The discriminant of π is:

$$\Delta = t^6[4a^2 + 27(c + dt^4)^3].$$

So by (e.g [17], §3) the fibration has a fiber of type I_0^* over $t = 0$, and over $t = \infty$ it has a fiber of type II^*. Observe that the action of σ on the holomorphic two-form $\omega_X = (dt \wedge dx)/2y$ is the multiplicity by ζ_8. This is a three dimensional family, so for the generic choice of the parameters of π the action of σ is trivial on Pic(X). Hence σ preserves each component of I_0^* , II^* and fixes respectively the component of multiplicity 2 and the component of multiplicity 6. On other hand, the curve C of genus $g(C) = 3$ meets I_0^* at three multiplicity one components and it cuts the fiber II^* in the isolated component of multiplicity 3 (since C intersects each fiber of π at three points (see Proposition 2.2.1)). Finally, by using Remark 2.1.11 one can find simply the local action at the 12 isolated points in $\mathrm{Fix}(\sigma)$.

Example 2.8.7. *The case* $m = 0$, $\mathrm{Pic(X)} = \mathrm{U} \oplus \mathrm{D}_4^{\oplus 2}$*:*

Consider the $K3$ surface X with elliptic fibration:

$$X : y^2 = x^3 + A(t)x + B(t),$$

where $A(t) = at^6 + bt^2$, $B(t) = ct^7 + dt^3$; $a, b, c, d \in \mathbb{C}$ with the non-symplectic order 8 automorphism $\sigma(x, y, t) = (-ix, \zeta_8 y, -it)$. Observe that σ acts as an order four automorphism on the basis of the fibration $\pi : X \longrightarrow \mathbb{P}^1$ and the two preserved fibers are over $t = 0$ and $t = \infty$.

The discriminant polynomial of π is:

$$\Delta(t) = 4A(t)^3 + 27B(t)^2 = t^6[4(at^4 + b)^3 + 27(ct^4 + d)^2].$$

By studying the zeros of $\Delta(t)$ and looking in the classification of singular fibers of elliptic fibrations on surfaces (e.g [17, §3]) one gets that: for a generic choice of the coefficients the fibration has two singular fibers G, G' of type I_0^* over $t = 0$ and $t = \infty$ respectively and it has 12 fibers of type I_1.

On the other hand, the number of moduli can be computed as follows: one can act on the equation by the transformation $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$ (and divide by λ^6) and by an automorphism of $\mathbb{P}^1$ given by a diagonal matrix. So one gets that the number of moduli is $4 - 1 - 1 = 2$. So that generically $\mathrm{rkT_X} = 12$ and $\mathrm{rkPic}(X) = 10$. Moreover by Shioda-Tate formula (rk Pic(X) $= 2 +$ rk MW $+ \sum_{\mathrm{F:fiber}}(\#\text{components of F} - 1)$) we get rk MW $= 0$, hence we can not have section of infinite order. This corresponds to the **eighth case of Table 2.4**. Observe that the curve C of genus 4 meets the two fibers G, G' in three multiplicity one components and the last multiplicity one component of I_0^* intersects the unique section E of π. Finally by studying carefully the action, one sees that the automorphism σ preserves each component of one of the two I_0^* fibers G, G' and exchanges two components in the other one (this corresponds to $l = 1$). It leaves invariant the component of multiplicity two and contains two fixed points on it of type $P^{4,5}$.
Using Remark 2.1.11 it is easy to find the local action at the 8 fixed points.

Example 2.8.8. Consider the elliptic fibration $\pi : X \longrightarrow \mathbb{P}^1$ in Weierstrass form given by:

$$y^2 = x^3 + A(t) + B(t),$$

where $A(t) = a_2 t^2 + a_6 t^6$ and $B(t) = b_3 t^3 + b_7 t^7 + b_{11} t^{11}$ with $a_2, a_6, b_3, b_7, b_{11} \in \mathbb{C}$. This fibration carries a non-symplectic order eight automorphism σ:

$$(x, y, t) \longmapsto (-ix, \zeta_8 y, -it).$$

The automorphism σ acts on the base of π as an automorphism of order four with two fixed points corresponds to $\sigma-$preserved fibers.

The discriminant polynomial of π is:

$$\Delta(t) = 4A(t)^3 + 27B(t)^2 = g_1 t^6 + g_2 t^{10} + g_3 t^{14} + g_4 t^{18} + g_5 t^{22}.$$

Such that:

$$g_1 = 4a_2^3 + 27b_3^2 \ , \ g_2 = 12a_6 + b_7 b_3.$$

$$g_3 = 12a_6^3 + 27b_7^2 + 54b_{11}b_3.$$

$$g_4 = 4a_6^3 + 54b_{11}b_7 \ , \ g_5 = 27b_{11}^2.$$

Observe (e.g by [17, §3]) that for a generic choice of the coefficients of $A(t)$ and $B(t)$ the fibration π has one fiber of type I_0^* over $t = 0$ and one fiber of type II over $t = \infty$ (this case has already appeared in Example 2.8.5).

On the other hand, if $g_1 = 0$ then the fibration π has a fiber of type I_4^* over $t = 0$ and a fiber of type II over $t = \infty$ and it has 12 fibers of type I_1. By doing the same argument as in the previous example on gets that $\mathrm{rkT_X} = 12$ (i.e $m_1 = 3$) and rk Pic(X) $= 10$. Moreover, by Shioda-Tate formula we get rk MW $= 0$ since the fibration has one reducible fiber of type I_4^* and the rank of $\mathrm{Pic}(X)$ is 10, this means that the fibration has a unique zero-section E. This section E is fixed by σ^4 and invariant with two isolated fixed points by σ and σ^2. Hence observe that $\mathrm{Fix}(\sigma^4)$ contains exactly 4 smooth rational curves, thus by [9, §4] see also ([4], Figure 1) the involution σ^4 fixes a smooth curve C of genus $g(C) = 5$. More precisely, the curve C meets with multiplicity three the fiber II at the singular point and the fiber I_4^* in three multiplicity one components, while the last component of multiplicity one of the I_4^* fiber intersects the unique section E of π. Finally by studying carefully the action of σ on the fibration π, one gets that the automorphism σ exchanges two components of I_4^* of multiplicity one that intersect C and σ leaves invariant the other components (so that we have $l = 1$). Using Remark 2.1.11 one obtains that $(n_2, n_3, n_4) = (4, 2, 2)$ such that $(k, h, N') = (1, 1, 2)$ in this case. So we have an example for the **seventh case of Table 2.4**.

If $\underline{g_1 = g_5 = 0}$ then by a generic choice of the coefficients the fibration admits a fiber of type I_4^* respectively of type I_0^* over $t = 0$ respectively $t = \infty$, and it has 8 fibers of type I_1 over the other zeros of $\Delta(t)$. In this case we have that the number of moduli is 1, so that generically $\mathrm{rkT_X} = 8$ and $\mathrm{rk\ Pic(X)} = 14$. Moreover, it follows by Shioda-Tate formula that rk M.W $= 0$ since $\mathrm{rk\ Pic(X)} = 14$ and the fibration has two reducible fibers one of them of type I_0^* and the other of I_4^* type, so that the fibration π has a unique zero-section E. On the other hand, observe that the involution σ^4 fixes exactly 5 smooth rational curves, thus by ([9], §4), see also [4, Figure 1], it fixes also a smooth curve of genus $g(C) = 2$. The curve C meets the two fibers of type I_4^* and I_0^* in three multiplicity one components and the last multiplicity one component intersects the zero section E. Finally by studying the action of σ on the fibration we get that the automorphism σ exchanges two components of multiplicity one in the fiber I_4^* respectively I_0^* that intersect C and σ leaves invariant the other components of each of this two fibers. Using Remark 2.1.11 one finds that $(n_2, n_3, n_4) = (3, 3, 4)$ and $(k, h, N') = ((1, 2, 2)$. So we get an example for the **last case in Table 2.4**.

Finally, if $\underline{g_1 = g_4 = g_5 = 0}$ then the fibration π has a fiber I_4^* over $t = 0$ and a fiber II^* over $t = \infty$. By the same argument one can get that $\mathrm{rk\ Pic(X)} = 18$ and π has a unique zero section E fixed by σ^4, moreover we get the involution σ^4 fixes a smooth elliptic curve C (in fact $\mathrm{Fix}(\sigma^4)$ contains 8 smooth rational curves and $\mathrm{rk\ Pic(X)} = 18$ so by ([9], §4) we have $g(C) = 1$). On the other hand, the elliptic curve C intersects the component of multiplicity three of the fiber II^* and meets fiber I_4^* in three multiplicity one components, while the zero section E intersects a component of multiplicity one in each one of this two fibers. By studying carefully the action of σ on the fibration π one sees that the automorphism σ preserves each component of the II^* fiber. On the other hand, σ exchanges two components that intersect C in the fiber of type I_4^* and σ leaves invariant the other components (this gives that $l = 1$ where we do not have a section of infinite order). By applying the Remark 2.1.11 we get $(n_2, n_3, n_4) = (6, 4, 4)$ so that $(k, h, N') = ((1, 2, 2)$ in this case. This is the **sixth case in Table 2.3**.

Example 2.8.9. *(Involution on the base of the fibration):*
Consider the elliptic fibration X given as:

$$y^2 = x(x^2 + tp_6(t))$$

with $p_6(t) := (a_6t^6 + a_4t^4 + a_2t^2 + a_0) = (t^2 - \alpha_1)(t^2 - \alpha_2)(t^2 - \alpha_3)$, and the order 8 non symplectic automorphism acting on it:

$$\sigma : (x, y, t) \mapsto (-ix, \zeta_8 y, -t).$$

The discriminant is $\Delta(t) = 27t^3(t^2 - \alpha_1)^3(t^2 - \alpha_2)^3(t^2 - \alpha_3)^3$. For generic choice of the coefficients the fibration has 8 fibers III (two tangent rational curves). This shows that rk Pic(X) ≥ 10. In fact by Shioda-Tate formula we have:

$$\text{rk Pic}(X) = 2 + \text{rk MW} + \sum_{F:\text{fiber}} (\#\text{components of } F - 1),$$

so we get rk Pic(X) $\geq 2 + \text{rk MW} + 8$ where the fibration has at least a zero section.

Observe that the fibration has a two torsion sections given by $t \mapsto (0:0:1) = (x:y:z)$ (while the zero section is $t \mapsto (0:1:0) = (x:y:z)$). Denote by τ the symplectic involution associated to this 2-torsion section. On the other hand, by [17, ch. I, §2] the J invariant of this elliptic fibration is $J = 4A(t)^3/(4A(t)^3 + 27B(t)^2) = 1$ (where $B(t) = 0$). This tells us that beside the fiber of type III (and the smooth fibers) the possible degenerations of the previous fibration (see [17, §3]) may produce singular fibers of type I_0^* and III^*. We will see this below.

The square of the automorphism σ, that is $\sigma^2 : (x, y, t) \mapsto (-x, iy, t)$, preserves each fiber and each fiber has an automorphism of order four. Moreover, σ^2 fixes two points on the generic smooth fiber, these two fixed points are contained in the 2-torsion section s_2 and in the zero section s_0. This gives that $k_{\sigma^2} \geq 2$.
The curve that cuts each fiber in $y = 0$ and $x^2 + tp_6(t) = 0$ has a $2:1$ morphism to $\mathbb{P}^1$ and has ramification points where $tp_6(t) = 0$, these correspond to the eight singular fibers III (over the seven zeros of $tp_6(t) = 0$ and one over infinity) and in fact C meets these fibers in their tangent point. The sections s_0 and s_2 meet one of the two components of III (not the same). We can find easily the genus of the curve $C \subset \text{Fix}(\sigma^4)$ by using Riemann-Hurwitz formula $2g(C) - 2 = -4 + 8$ so that $g(C) = 3$. In fact recall that the Riemann-Hurwitz formula applied to a 2:1 morphism from a curve C to the projective line $\mathbb{P}^1$ is given by:

$$2g(C) - 2 = 2(0 - 2) + \sum_{p \in C} (e_p - 1),$$

where e_p is the ramification index at a ramification point p which in this case is $e_p = 2$. Observe moreover that the curve C contains two fixed points for the action of σ thus $N' = 2$, and since σ preserves the two sections s_0 and s_2 and has two fixed points on it then $N = 6$. Looking at [2, Example 8.2] we see that in this case Pic(X) $= \mathrm{U}(2) \oplus \mathrm{D}_4^{\oplus 2}$, so we are in **case 9 in Table 2.4**. Observe that moreover the family of such elliptic fibrations is 2-dimensional, so this is the whole family (since $m_1 = 3$). If we compose σ with τ the two sections are exchanged so that C still contains two isolated fixed points, hence $N' = 2$ and also $N = 2$. This is **case 10 in Table 2.4**.
We start now to consider degenerations of the previous family. Remark that in the following cases the situation is as before for the action of σ and $\sigma \circ \tau$ on the generic fiber.

- $\alpha_1 = \alpha_2$ (and similar cases):
  If $\alpha_1 = \alpha_2$, then the discriminant of the fibration is given by:

  $$\Delta(t) = 27t^3(t^2 - \alpha_1)^6(t^2 - \alpha_3)^3.$$

  This gives two fibers I_0^* exchanged by σ and 4 fibers of type III, two of them over 0 and ∞ (observe that we can not get the two fibers I_0^* on 0 and ∞), thus by Shioda-Tate formula we have rk Pic(X) ≥ 14. On the other hand, the genus of the curve C in this case is $2g(C) - 2 = -4 + 4$ (by using Riemann-Hurwitz formula) so that $g(C) = 1$. Remark that the situation is as before for the action of σ and $\sigma \circ \tau$ on C. Hence either (N, N') equals $(6, 2)$ or $(2, 2)$ and so we get the **cases 3 and 4 in Table 2.3**.

- $\alpha_1 = 0$ (and similar cases):
In this case the fibration has a fiber III^* : $R_1+2R_2+3R_3+4R_4+2R_5+3R_6+2R_7+R_8$ over $t=0$ and a fiber of type III over ∞ and four more fibers of type III over the other zeros of $\Delta(t)$. Here $g(C)=2$ and rk Pic(X) ≥ 14 (by doing the same previous argument). Moreover, observe that the curve C meets the III^* fiber in R_5 and the sections s_0, s_2 meets this fiber in one of the two components R_1, R_8 (not the same). So that $N'=2$ for the automorphism σ and by looking at the possible action at the intersection points of the components of the III^* fiber we get that $N=10$. On the other hand, for $\sigma\circ\tau$ the curve C still contains two fixed points thus $N'=2$ while $N=4$ in this case. We get examples for the **cases 15 and 16 of Table 2.4**.

- $\alpha_1=\alpha_2=0$ (and similar cases): Here we have a fiber III^* over zero and over ∞, and two fibers III over the other zeros of $\Delta(t)$. We have $g(C)=1$. We get here examples for the **cases 6 and 7 of Table 2.3**.

- $\alpha_1 = 0$ and $\alpha_2=\alpha_3$ (and similar cases): Here we have a fiber of type III^* over zero and a fiber III over ∞ and we have two fibers I_0^* exchanged by σ. Here we have $2g(C)-2=-4+2=-2$ so that $g(C)=0$ (i.e the all fixed curves by σ^4 are rational) and then the curve C contains two fixed points (one of them of type $P^{2,7}$ and the other of type $P^{3,6}$). Hence for σ we find $k=1$ and $N=10$, this is the **only case in Theorem 2.6.1**. For $\sigma\circ\tau$ we find $k=0$ and $N=4$ with $(n_4,n_2,n_3)=(2,1,1)$. We have moreover that $k_{\sigma^2}=3$ and $N_{\sigma^2}=10$ this is **the last case in Table 2.8**.

Example 2.8.10. *Double cover of the quadric:*
Consider the following automorphism of $\mathbb{P}^1\times\mathbb{P}^1$:

$$j:((x_0,x_1),(y_0,y_1))\mapsto((x_0,ix_1),(y_0,iy_1))$$

The double cover of $\mathbb{P}^1\times\mathbb{P}^1$ ramified along a curve of bidegree $(4,4)$ is a K3 surface. Take the equation of the curve $f(x_0,x_1,x_2,x_3)$ such that $j(f)=if$ so that the double cover $w^2=f$ is invariant by the automorphism:

$$\sigma:((x_0,x_1),(y_0,y_1),w)\mapsto((x_0,ix_1),(y_0,iy_1),\zeta_8 w)$$

The equation of f is:

$$y_0^3y_1(a_0x_0^4+a_1x_1^4)+x_0^3x_1(b_0y_0^4+b_1y_1^4)+x_0x_1^2y_0y_1^2(c_0x_1y_0+c_1x_0y_1)$$

with some parameters $a_0,a_1,b_0,b_1,c_0,c_1\in\mathbb{C}$. Observe that by applying transformations of $\mathbb{P}^1\times\mathbb{P}^1$ we see that the family depends in fact on 3 parameters $(6-1-2=3)$. The double cover is a K3 surface and the fixed locus of σ^4 is a curve of genus 6. There is a D_4 singularity in $((0:1),(0:1),0)$. In fact consider the embedding in $\mathbb{P}^3$ by Segre:

$$((x_0:x_1),(y_0:y_1))\mapsto(x_0y_0:x_0y_1:x_1y_0:x_1y_1)=(x:y:z:t)$$

The image is the quadric $xt=yz$ and the singular point of the image of $C=\{f=0\}$ is $(0:0:0:1)$. The image of C on $xt=yz$ is:

$$a_0x^3y+a_1z^3t+b_0x^3z+b_1y^3t+c_0ytz^2+c_1xyt^2=0.$$

In the chart $t = 1$ the equation of the quadric becomes $x = yz$ and the equation of C is hence:

$$a_0x^3y + a_1z^3 + b_0x^3z + b_1y^3 + c_0yz^2 + c_1xy = 0.$$

Choosing local coordinates (y, z) on the quadric (the partial derivative with respect to x is equal to one, so one can use implicit function theorem). We get an equation:

$$a_0y^4z^3 + b_0y^3z^4 + a_1z^3 + b_1y^3 + c_0yz^2 + c_1y^2z = 0.$$

Using the classification of [22, Ch. 2 §8] one obtains that the singularity is a D_4 singularity. Observe that in the charts $x = 1$, $y = 1$, $z = 1$ there are no singularities (one can do the computation by hand or use the computer algebra system MAGMA).
By computing the number of moduli one gets that $m_1 = 4$ thus rk $\text{Pic}(X) = 6$. So that the Picard group of the double cover is $U(2) \oplus D_4$. We have four fixed points on the singular curve C: the singular point $((0 : 1), (0 : 1))$ and the three smooth points $((0 : 1), (1 : 0))$, $((1 : 0), (0 : 1))$, $((1 : 0)(1 : 0))$ this tells us immediately that the number of fixed points on the smooth curve corresponding to C on the desingularization X of the double cover contains at least 3 isolated fixed points. This is the **case 4 in Table 2.4**.

Example 2.8.11. *Quadruple Quartics:*
Take the fourfold cover of $\mathbb{P}^2$:

$$t^4 = x_0(l_3(x_1, x_2) + x_0^2 l_1(x_1, x_2))$$

where $l_3(x_1, x_2)$ is homogeneous of degree three and $l_1(x_1, x_2)$ is homogeneous of degree 1. This is invariant for the action of the order 8 non symplectic automorphism:

$$(t, x_0, x_1, x_2) \mapsto (\zeta_8 t, -x_0, x_1, x_2)$$

it fixes the inverse image of the curve $\{x_0 = 0\}$ which is rational and 4 points on the curve $C : \{l_3(x_1, x_2) + x_0^2 l_1(x_1, x_2) = 0\}$ which is in fact elliptic. This gives another example for the **case 4 of Table 2.2**.

Chapter 3

Non-symplectic automorphism of order 16

In this chapter we classify $K3$ surfaces with non-symplectic automorphism of order 16 in full generality. We obtain a complete classification for the non-symplectic automorphisms of order 16 based on the classification of non-symplectic automorpfisms of order 4 on $K3$ surfaces given by [2], without the use of our results contained in Ch 2, following the submitted paper [1].

3.1 The fixed locus.

Let X be a $K3$ surface and σ a *non-symplectic* automorphism of order 16 acting on it, this means that the action of σ^* on the vector space $H^{2,0}(X) = \mathbb{C}\omega_X$ of holomorphic two-forms is not trivial. More precisely we assume that the automorphism σ is *(purely)* non-symplectic, i.e. $\sigma^*\omega_X = \zeta_{16}\omega_X$, where ζ_{16} is a primitive 16th root of unity. For simplicity we omit sometimes "purely".

We denote furthermore by $r_{\sigma^j}, l_{\sigma^j}, m_{\sigma^j}$, $m^1_{\sigma^j}$, $m^2_{\sigma^j}$ for $j = 1,2,4,8$ the rank of the eigenspaces of $(\sigma^j)^*$ in $H^2(X,\mathbb{C})$ relative to the eigenvalues $1, -1$, i, ζ_{16}^2 and ζ_{16}. For simplicity for $j = 1$ we just write r_σ, $l_\sigma, \ldots$ or even $r, l, \ldots$.

Remark 3.1.1. • *Doing a simple computation we get the following relations:*

$$\begin{aligned} r_{\sigma^2} &= r_\sigma + l_\sigma, & l_{\sigma^2} &= 2m_\sigma, \quad m_{\sigma^2} = 2m^1_\sigma, \quad m^1_{\sigma^2} = 2m^2_\sigma, \\ r_{\sigma^4} &= r_\sigma + l_\sigma + 2m_\sigma, & l_{\sigma^4} &= 4m^1_\sigma, \quad m_{\sigma^4} = 4m^2_\sigma, \\ r_{\sigma^8} &= r_\sigma + l_\sigma + 2m_\sigma + 4m^1_\sigma, & l_{\sigma^8} &= 8m^2_\sigma, \\ r_\sigma + l_\sigma &+ 2m_\sigma + 4m^1_\sigma + 8m^2_\sigma = 22. \end{aligned}$$

• *As a direct consequence of the previous relations one can get immediately that the invariants $l_{\sigma^2}, m_{\sigma^2}, m^1_{\sigma^2} \in 2\mathbb{Z}$ while the invariants $l_{\sigma^4}, m_{\sigma^4} \in 4\mathbb{Z}$.*

We recall the invariant lattice

$$S(\sigma^j) = \{x \in H^2(X,\mathbb{Z}) \,|\, (\sigma^j)^*(x) = x\},$$

and its orthogonal

$$T(\sigma^j) = S(\sigma^j)^\perp \cap H^2(X,\mathbb{Z}),$$

clearly we have that $\operatorname{rk} S(\sigma^j) = r_{\sigma^j}$.

By [8, Theorem 3.1] the eigenvalues of the action of σ on T_X are primitive 16th roots of unity so $\operatorname{rk}(T_X) = 8m_\sigma^2$ (see §1.2.2, Proposition 1.2.13). Since $0 < \operatorname{rk}(T_X) \leq 21$ we have in fact only two possibilities which are $m_\sigma^2 = 1$ or 2 so that $\operatorname{rk}\operatorname{Pic}(X) = 14$ respectively 6. As we have remarked in §1.2.2 in the generic case this is also the rank of $S(\sigma^8) = \operatorname{Pic}(X)$ and we have by orthogonality $T_X = T(\sigma^8)$. Recall moreover that $r_\sigma > 0$, see Proposition 1.2.11, § 1.2.2. We start recalling the classifiction theorem for non-symplectic involution on $K3$ surfaces (see Theorem 1.2.17 in Ch 1 , [9, Theorem 4.2.2] and also [11, §4]).

Theorem 3.1.2. *Let τ be a non-symplectic involution on a K3 surface X. The fixed locus of τ is either empty, the disjoint union of two elliptic curves or the disjoint union of a smooth curve of genus $g \geq 0$ and k smooth rational curves.*

Moreover, its fixed lattice $S(\tau) \subset \operatorname{Pic}(X)$ is a 2-elementary lattice with determinant 2^a such that:

- *$S(\tau) \cong U(2) \oplus E_8(2)$ iff the fixed locus of τ is empty;*
- *$S(\tau) \cong U \oplus E_8(2)$ iff τ fixes two elliptic curves;*
- *$2g = 22 - \operatorname{rk} S(\tau) - a$ and $2k = \operatorname{rk} S(\tau) - a$ otherwise.*

At a fixed point for σ^j the action can be linearized (see § 1.2.2 and e.g. [8, §5]) and is given by a matrix

$$A_{t,s}^j = \begin{pmatrix} \zeta_{(16/j)}^t & 0 \\ 0 & \zeta_{(16/j)}^s \end{pmatrix}$$

with $t + s = 1 \mod (16/j)$, $0 \leq t < s < 16/j$. This means that the fixed locus of σ^j is the disjoint union of smooth curves and isolated points (see [9, Section 4, §2] and [8, §5]). In the sequel of this chapter when we consider curves in the fixed locus of some σ^j we always mean smooth curves. By Hodge index theorem $\operatorname{Fix}(\sigma^j)$ may contains only one curve of genus $g > 1$. We denote by k_{σ^j} the number of fixed rational curves, by N_{σ^j} the number of fixed points in $\operatorname{Fix}(\sigma^j)$. Moreover by $n_{t,s}^{\sigma^j}$ we denote the number of isolated fixed points of type $P^{t,s}$ by σ^j. In several cases when it is clear which automorphism we are considering we just write k, N, $n_{t,s}$, and so on.

Lemma 3.1.3. *Let σ be a non-symplectic automorphism of order 16 acting on a $K3$ surface X and let a_{σ^4} be the number of rational curves interchanged by σ^4 and fixed by σ^8, then $a_{\sigma^4} \in 4\mathbb{Z}$.*

Proof. A curve as in the statement has stabilizer group in $\langle\sigma\rangle$ of order 2. Hence its σ-orbit has length 8, so we get that a_{σ^4} is a multiple of 4. □

We denote by $2a$ the number of exchanged smooth rational curves by σ and fixed by σ^2, and by $8\bar{s}$ the number of smooth rational curves cyclic permuted by σ and fixed by σ^8 (clearly they are permuted by σ^2, four by four, and they are interchanged by σ^4 two by two).

We formulate now Proposition 3.1.4 that we need to prove Theorem 3.1.8. We show then in Proposition 3.2.3 that the case $g(C) = 1$ is not possible.

Proposition 3.1.4. *Let σ be a non-symplectic automorphism of order 16 acting on a $K3$ surface X with $S(\sigma^8) \cong Pic(X)$. If $C \subset \operatorname{Fix}(\sigma)$ then $g(C) = 0, 1$, and we can not have two curves of genus one in the fixed locus.*

Proof. If $C \subset \text{Fix}(\sigma)$ then this is also fixed by σ^4 which is non-symplectic of order 4. If $g(C) \geq 2$ by the relations (3.1.1) we have that l_{σ^4} and m_{σ^4} are multiples of 4, checking in [2, Theorem 4.1] the only possible case is $(m_{\sigma^4}, r_{\sigma^4}, l_{\sigma^4}) = (4, 6, 8)$ and $N_{\sigma^4} = 2$, $k_{\sigma^4} = 0$, $g(C) = 2$. By the classification of Nikulin (see [11, §4]) the involution σ^8 fixes five rational curves other than the curve of genus 2. Since $k_{\sigma^4} = 0$, four of the rational curves are interchanged two by two by σ^4, one rational curve is preserved and contains the two fixed points. In this case $a_{\sigma^4} = 2$ contradicting Lemma 3.1.3. If $g(C) = 1$ and there exists another genus one curve $C' \subset \text{Fix}(\sigma)$, then by Theorem 3.1.2 $\text{rk}\, S(\sigma^8) = 10$ but this is not possible, since the rank can be only equal to 6 or 14 as we have explained above. □

Remark 3.1.5. *More in general by the same reason as in Proposition 3.1.4 if* $\text{Fix}(\sigma^8)$ *contains an elliptic curve then this is the only one. We exclude also the case of* $\text{Fix}(\sigma^8) = \emptyset$ *(here again is* $\text{rk}\, S(\sigma^8) = 10$ *and this is not possible). The fact that* $\text{Fix}(\sigma^j) \neq \emptyset$*,* $j = 1, 2, 4$ *follows immediately from the holomorphic Lefschetz formula, indeed the Lefschetz number is not zero (see Proposition 3.1.8, Proposition 3.1.11 and [2, Proposition 1]).*

We recall now Lemma 1.2.15 and the following useful remark which is a direct application of the Remark 1.2.16 when the order of a non-symplectic automorphism is $n = 16$ (see also e.g. [2, Lemma 4]):

Lemma 3.1.6. *Let* $T = \sum_h R_h$ *be a tree of smooth rational curves on a* $K3$ *surface* X *such that each* R_h *is invariant under the action of a non-symplectic automorphism* η *of order* j*. Then, the points of intersection of the rational curves* R_h *are fixed by* η *and the action at one fixed point determines the action on the whole tree.*

Remark 3.1.7. *In the case of an automorphism of order* 16*, with the assumption of Lemma 1.2.15, the local actions at the intersection points of the curves* R_i *appear in the following order (we give only the exponents of* ζ *in the matrix of the local action):*

$$\ldots, (0,1), (15,2), (14,3), (13,4), (12,5), (11,6), (10,7), (9,8),$$
$$(8,9), (7,10), (6,11), (5,12), (4,13), (3,14), (2,15), (1,0), \ldots$$

This remark will be particularly useful when we study elliptic fibrations on X*.*

Proposition 3.1.8. *Let* σ *be a non-symplectic automorphism of order* 16 *acting on a* $K3$ *surface* X*. Then the fixed locus is non-empty and*

$$\text{Fix}(\sigma) = C \cup E_1 \cup \cdots \cup E_k \cup \{p_1, \cdots, p_N\}$$

or

$$\text{Fix}(\sigma) = E_1 \cup \cdots \cup E_k \cup \{p_1, \cdots, p_N\},$$

where C *is a curve of genus* $g = 1$*, the* E_i*'s are rational fixed curves,* $k = k_\sigma$ *and the* p_i*'s are isolated fixed points,* $N = N_\sigma$*. Moreover* N *is even,* $4 \leq N \leq 16$ *and the following relations hold :*

$$N = n_{3,14} + n_{4,13} + n_{5,12} + n_{6,11} + 2n_{7,10} + 2k + 1. \tag{I}$$

$$N = 2n_{3,14} + 2n_{5,12} + 2n_{7,10} + 2k. \tag{II}$$

$$N = 2 + r_\sigma - l_\sigma - 2k. \tag{III}$$

Proof. By Proposition 3.1.4 we know that the fixed locus may contain at most one curve of genus one. We use first the topological Lefschetz fixed point formula for σ. We write $r = r_\sigma$ and $l = l_\sigma$. We have

$$\begin{aligned} N + \textstyle\sum_{K\subset \mathrm{Fix}(\sigma)}(2-2g(K)) &= \chi(\mathrm{Fix}(\sigma)) = \textstyle\sum_{h=0}^{4}(-1)^h \operatorname{tr}(\sigma^*|H^h(X,\mathbb{R})) \\ &= 2 + \operatorname{tr}(\sigma^*|H^2(X,\mathbb{R})). \end{aligned}$$

This gives $N + 2k = \chi(\mathrm{Fix}(\sigma)) = r - l + 2$ so that $r - l = N + 2k - 2$ (this gives (III)). Since $\operatorname{rk} S(\sigma) = 14$ or 6 in any case we have $N \le 16$. We use now holomorphic Lefschetz formula (see [24, Theorem 4.6]). The Lefschetz number is

$$L(\sigma) = \sum_{h=0}^{2}(-1)^h \operatorname{tr}(\sigma^*|H^h(X,\mathcal{O}_X)) = 1 + \zeta_{16}^{-1},$$

on the other hand

$$L(\sigma) = \sum_{t,s}\frac{n_{t,s}}{\det(I-\sigma^*|T_x)} + \frac{1+\zeta_{16}}{(1-\zeta_{16})^2}\sum_{K\subset\mathrm{Fix}(\sigma)}(1-g(K))$$

where T_x denotes the tangent space at an isolated fixed point x. Using the expression for the local action of σ at x and comparing the two expressions for $L(\sigma)$ we get the equations:

$$n_{2,15} - n_{7,10} + n_{8,9} = 1 + 2k. \tag{3.1.1}$$

$$n_{2,15} - n_{3,14} + n_{4,13} - n_{5,12} + n_{6,11} - n_{7,10} + n_{8,9} = 2k. \tag{3.1.2}$$

$$n_{4,13} + n_{5,12} - 2n_{6,11} + 2n_{7,10} - n_{8,9} = 2k. \tag{3.1.3}$$

$$2n_{3,14} - 2n_{4,13} + 2n_{6,11} - n_{8,9} = 2k. \tag{3.1.4}$$

Combining (3.1.1) and (3.1.2) we get

$$n_{3,14} - n_{4,13} + n_{5,12} - n_{6,11} = 1. \tag{3.1.5}$$

From (3.1.1) and (3.1.2) and the fact that $N = \sum n_{t,s}$ we obtain the relations (I) and (II) in the statement respectively. By (I) we get that $N \ge 1$ and by (II) we find that N is an even number, thus $N \ge 2$. If $N = 2$ then by (I) we obtain $k = n_{7,10} = 0$ and either $n_{3,14}$ or $n_{5,12}$ is equal to 1 by relations (I) and (II) , thus $n_{4,13} = n_{6,11} = 0$ by (I) and either $n_{2,15}$ or $n_{8,9}$ is equal to one by (3.1.1). By (3.1.4) we obtain $n_{8,9} = 2n_{3,14}$ so $n_{8,9} = n_{3,14} = 0$. By using (3.1.3) we obtain $n_{5,12} = 0$ which is impossible. So $N \ge 4$. □

Remark 3.1.9. [1] *As a direct consequence of formulas in Proposition 3.1.8 we find :*

- *If* $N = 4$ *we have only the possibility with* $(n_{3,14}, n_{7,10}, n_{8,9}, k) = (1, 1, 2, 0)$ *(the other* $n_{t,s}$ *are zero) so that* $r - l = 2$.
- *The case* $(N, k) = (8, 0)$ *is not possible.*
- *If* $(N, k) = (6, 0)$ *then* $(n_{5,12}, n_{6,11}, n_{7,10}, n_{8,9}) = (2, 1, 1, 2)$*, the other* $n_{t,s}$ *are zero and* $r - l = 4$.
- *If* $(N, k) = (6, 1)$ *then* $(n_{2,15}, n_{3,14}, n_{7,10}) = (4, 1, 1)$*, the other* $n_{t,s}$ *are zero and* $r - l = 6$.

[2] *The fixed points for σ with local action $(2,15)$, $(7,10)$, $(3,14)$, $(6,11)$, are isolated fixed points for σ^4, whence the points of type $P^{8,9}$, $P^{4,13}$ and $P^{5,12}$ are contained in a fixed curve for σ^4. The points of type $P^{8,9}$ are contained in a fixed curve for σ^2.*

Proposition 3.1.10. *Let σ be a purely non-symplectic automorphism of order 16 acting on a $K3$ surface X. The fixed locus $\mathrm{Fix}(\sigma^4)$ contains at least one fixed curve C of genus 0 or 1 (and no curves of higher genus).*

Proof. If $\mathrm{Fix}(\sigma^4)$ contains only isolated fixed points then by Remark 3.1.9 we have $n_{4,13} = n_{5,12} = n_{8,9} = k = 0$. By equation (3.1.4) we obtain $n_{3,14}+n_{6,11} = 0$ so they are both equal to 0. We get a contradiction to equation (3.1.5). Finally if the order four automorphism σ^4 fixes a curve C of genus $g(C) > 1$ and since by Remark 3.1.1 we have $l_{\sigma^4}, m_{\sigma^4} \in 4\mathbb{Z}$, we get $(m_{\sigma^4}, r_{\sigma^4}, l_{\sigma^4}) = (4, 6, 8)$ by [2, Theorem 4.1]. In this case $a_{\sigma^4} = 2$ which contradicts Lemma 3.1.3. □

In the following proposition and remark we recall the relations for the number of fixed points and curves by a non-symplectic automorphism of order eight σ^2, and the local action of σ^2 at the intersection points of tree of smooth rational curves that appearing in Proposition 2.1.8 respectively Remark 2.1.11. We give also some other results.

Proposition 3.1.11. *Let σ be a non-symplectic automorphism of order 16 on a $K3$ surface X and $C \subset \mathrm{Fix}(\sigma^2)$. Then $g(C) \leq 1$ and the following relations for the number of fixed points and curves by σ^2 hold:*

$$\begin{array}{lcl} n_{2,7} + n_{3,6} & = & 2 + 4k_{\sigma^2}, \\ n_{4,5} + n_{2,7} - n_{3,6} & = & 2 + 2k_{\sigma^2}, \\ N_{\sigma^2} & = & 2 + r_{\sigma^2} - l_{\sigma^2} - 2k_{\sigma^2}, \end{array}$$

where $n_{t,s}$ denote the number of fixed points of type (t,s) for the action of σ^2.

Proof. Observe that by Proposition 3.1.10 we have $g(C) \leq 1$ moreover an isolated fixed point for σ^2 is given by the local action $\begin{pmatrix} \xi^t & 0 \\ 0 & \xi^s \end{pmatrix}$, $t+s = 1 \mod (8)$, $0 \leq t < k < 8$. We obtain the relations in the statement by applying holomorphic and topological Lefschetz's formulas (see Proposition 2.1.8). □

Remark 3.1.12.

- *By Lemma 1.2.15, and with the same notation there, the local action of σ^2 at the intersection points of the curves R_h appear in the following order:*

$$\ldots, (0,1), (7,2), (6,3), (5,4), (4,5), (3,6), (2,7), (1,0), \ldots$$

- *The σ-fixed points of type $P^{5,12}$ and $P^{4,13}$ give σ^2-fixed points of type $P^{4,5}$, the σ-fixed points of type $P2,15$ and $P^{7,10}$ give σ^2-fixed points of type $P^{2,7}$ (up to the order). The σ-fixed points of type $P^{3,14}$ and $P^{6,11}$ give σ^2-fixed points of type $P^{3,6}$ (up to the order).*

3.2 The case of an invariant elliptic curve.

Here we suppose that the involution σ^8 fixes an elliptic curve C. Thus, as we have seen in Section 2.3, the $K3$ surface X carries an elliptic fibration $\pi_C : X \longrightarrow \mathbb{P}^1$ having C as a smooth fiber. The fibration π_C is invariant by σ^i ; $i = 1, 2, 4, 8$ (since σ^i preserves C which is a fiber of π_C) and all curves fixed by σ^i are contained in the fibers of π_C, that because they are disjoint with C and the action on the base of σ is non-trivial. In fact if the action would be trivial then a smooth fiber would have an automorphism of order 8. An elliptic curve can admit only automorphisms of order 2,4,6 (different from a translation), so that this automorphism should be induced by a translation by a point of order 8 on the generic fiber. But then σ would be a symplectic automorphism, which contradicts our assumption on σ.

Lemma 3.2.1. *If X carries a $\sigma-$invariant elliptic fibration, such that σ^8 fixes an irreducible smooth fiber C of this fibration, then σ acts with order 16 on the basis of the fibration and fixes two points on it.*

Proof. The proof is given as the same way as in Lemma 2.3.1 of Section 2.3. □

We recall first some notations. We denote by k_{σ^j} the number of fixed rational curves, by N_{σ^j} for $j = 1, 2, 4, 8$ the number of fixed points in Fix(σ^j), and by $2a_{\sigma^4}$ the number of exchanged smooth rational curves by σ^4 and fixed by σ^8.

Theorem 3.2.2. *Let σ be a purely non-symplectic automorphism of order 16 on a $K3$ surface X with Pic(X) = $S(\sigma^8)$ and C be an elliptic curve in Fix(σ^8). Then σ acts as an automorphism of order four on C and we have the following cases*

m^2	m^1	m	l	r	N	k	type of C'
1	1	0	1	9	8	1	IV*
1	1	0	3	7	6	0	IV*

Table 3.1: The case $g(C)$=1

Proof. Since σ preserves C, then there is a $\sigma-$invariant elliptic fibration $\pi_C : X \longrightarrow \mathbb{P}^1$ with a generic fiber C. Observe that by Lemma 3.2.1 the automorphism σ has order sixteen on the basis of π_C and it has two fixed points on $\mathbb{P}^1$, corresponding to the fiber C and a fiber C' of π_C. This implies that all rational curves fixed by σ are contained in C'.

At first observe that the elliptic curve $C \not\subseteq \text{Fix}(\sigma)$. In fact if $C \subset \text{Fix}(\sigma)$ then the fibration π_C admits also an automorphism of order four σ^4, since by Remark 3.1.1 we have $l_{\sigma^4}, m_{\sigma^4} \in 4\mathbb{Z}$, by [2, Theorem 3.1] we get $(m_{\sigma^4}, r_{\sigma^4}, l_{\sigma^4}) = (4, 10, 4)$ and the fixed locus of σ^4 contains 1 rational fixed curve and 6 isolated fixed points (here $a_{\sigma^4} = 0$), we get moreover that the fiber C' is of type IV^* in this case.
The component of multiplicity 3 in the fiber IV^* is clearly σ-invariant. If it is fixed by σ then each other component is preserved, so that $k = 1$ and $N = 6$. More precisely by Remark 3.1.7 we have $n_{2,15} = n_{3,14} = 3$ which contradicts Remark 3.1.9. If the component of multiplicity 3 is σ-invariant then it contains 2 isolated fixed points. Two branches of the fiber are exchanged and we have $N = 4$. By Remark 3.1.9 we have $n_{8,9} = 2$, $n_{7,10} = 1$, $n_{3,14} = 1$ but this is not possible by using the Remark 3.1.7.

Assume now that the curve $C \subset \mathrm{Fix}(\sigma^4)$. As we have seen previously that σ fixes two points on the basis of π_C corresponding to the elliptic curve C and a singular fiber C' of type IV^*, and we are in the case $(m_{\sigma^4}, r_{\sigma^4}, l_{\sigma^4}) = (4, 10, 4)$ with $(k_{\sigma^4}, N_{\sigma^4}, a_{\sigma^4}) = (1, 6, 0)$ by [2, Theorem 3.1]. Clearly σ leaves invariant the fiber C and the singular fiber C' of type IV^*. The latter corresponds to the other fixed point for the action of σ on the base $\mathbb{P}^1$. By the previous remark the curve C can not be fixed by σ, hence σ has order 2 or 4, with fixed points, on it or it is a translation. By basic results on automorphisms on elliptic curves, in the first two cases σ fixes four, respectively two points on C. There are two possible actions on C'.

First case: The singular fiber of type IV^* contains a fixed rational curve, which is necessarily the component of multiplicity 3. Then by using the Lemma 3.1.6 and the formulas in Proposition 3.1.8 we find $k = 1$, $N = 8$ with $n_{2,15} = n_{3,14} = 3$ and $n_{4,13} = 2$ the other $n_{t,s}$ are zero. In particular σ must have two fixed points on C this means that it acts as an automorphism of order four.

Second case: The singular fiber of type IV^* has a reflection of order 2. Then the curve of multiplicity 3 is preserved and contains two isolated fixed points with action $(8, 9)$. In fact this curve must be fixed by σ^2 otherwise it would contain too many isolated fixed points for the action of σ^2. Combining Remark 3.1.9 and Proposition 3.1.8 we find $(N, k) = (6, 0)$, with $n_{8,9} = 2 = n_{5,12}$, $n_{7,10} = 1 = n_{6,11}$, the other $n_{t,s}$ are zero. We observe that also in this case σ must have two fixed points on C, this means that it acts as an automorphism of order four.

Using the fact that $(m_{\sigma^4}, r_{\sigma^4}, l_{\sigma^4}) = (4, 10, 4)$ we get immediately that in both cases $m^2_\sigma = m^1_\sigma = 1$. Moreover we have that $r_\sigma + l_\sigma + 2m_\sigma = 10$ and in the first case we have $r - l = 8$, in the second case $r - l = 4$. In both cases we have $N_{\sigma^2} = 10$ and $k_{\sigma^2} = 1$ so using Proposition 3.1.11 we obtain the values of r, l, m given in the table. In this two cases we have that $\mathrm{rk}\ \mathrm{Pic}(X) = \mathrm{rk}\ S(\sigma^8) = 14$.

Finally, assume that $C \not\subseteq \mathrm{Fix}(\sigma^4)$. By the equations of Remark 3.1.1 and since $k_{\sigma^4} > 0$ by Proposition 3.1.10, we are in the case $(m_{\sigma^4}, r_{\sigma^4}, l_{\sigma^4}) = (4, 10, 4)$ with $(k_{\sigma^4}, N_{\sigma^4}, a_{\sigma^4}) = (1, 6, 0)$ in [2, Theorem 8.4]. The order 4 automorphism σ^4 acts on the curve C as a translation and the six fixed points by it are contained in the reducible fiber C' of type IV^*. This case is not possible by the same reason of the first case in this proof (where σ acts on the curve C as a translation with no fixed points on it). □

Proposition 3.2.3. *Let σ be a purely non-symplectic automorphism of order 16 acting on a $K3$ surface X. If $C \subset \mathrm{Fix}(\sigma)$ then C is rational.*

Proof. By Proposition 3.1.4 we have that if σ fixes a curve C, then C is either smooth elliptic curve or rational. The case when $g(C) = 1$ is not possible by Theorem 3.2.2 (see the first case in the proof of the previous theorem). □

3.3 The rank six case.

In this section we study the case when $\mathrm{Pic}(X) = S(\sigma^8)$ has rank 6. We denote here by N' the number of fixed points that are contained in the curve $C \subset \mathrm{Fix}(\sigma^8)$, by $2h$ the number of interchanged points by σ on it. Observe that the fixed points by σ on C are of type $P^{7,10}, P^{2,15}, P^{3,14}$ and $P^{6,11}$. In fact by Proposition 3.1.10 we get if C is fixed by the order four automorphism σ^4 then $g(C) = 0, 1$, and by Theorem 3.2.2 the rank of $\mathrm{Pic}(X)$ is 14 when $g(C) = 1$. We denote also by k_{σ^j} the number of fixed rational curves, by N_{σ^j} for $j = 1, 2, 4, 8$ the number of fixed points in $\mathrm{Fix}(\sigma^j)$, and by $2a_{\sigma^4}$ the number of exchanged smooth rational curves by σ^4 and fixed by σ^8.

The following proposition will be useful in the proof of Theorem 3.3.2.

Proposition 3.3.1. *Let σ be a non-symplectic automorphism of order 16 on a $K3$ surface X such that $\mathrm{Pic}(X) = S(\sigma^8) \cong U \oplus L$ where L is isomorphic to a direct sum of root lattices of types A_1, D_{4+n}, E_7 or E_8 and σ^8 fixes a curve of genus $g > 1$. Then X carries a jacobian elliptic fibration $\pi : X \longrightarrow \mathbb{P}^1$ whose fibers are σ^8-invariant and it has reducible fibers described by L and a unique section $E \subset \mathrm{Fix}(\sigma^8)$. Moreover , if $g > 4$ then π is $\sigma-$invariant .*

Proof. Since $\mathrm{Pic}(X) = S(\sigma^8) \cong U \oplus L$ the first half of the statement follows from [5, Lemma 2.1, 2.2]. On other hand, since σ^8 fixes a curve C of genus $g > 1$, then C intersects each fiber of π in at least two points. This implies that σ^8 preserves each fiber of π and acts on it as an involution with four fixed points. By [6, Theorem 6.3] we have that the Mordell-Weil group of π is $MW(\pi) \cong \mathrm{Pic}(X)/T$ where T denote the subgroup of $\mathrm{Pic}(X)$ generated by the zero section and fiber components. Since L is a root lattice and $\mathrm{Pic}(X) \cong U \oplus L$ we have that $MW(\pi)$ is trivial, hence π has a unique section E. Since σ^8 preserves each fiber of π and E is invariant, we have that E is fixed by σ^8. This implies that C intersects each fiber in three points and one fixed point for the action of σ^8 is contained in the section E.

Now we will prove that π is $\sigma-$invariant if $g > 4$. Let f be the class of a fiber of π. The automorphism σ preserves the curve C, and we have that $CE = 0$ (the fixed curves for σ^8 can not intersect). Assume that $f \neq \sigma^*(f)$ then they intersect in at least 2 points. Indeed if $f\sigma^*(f) = 1$ then this is a fixed point of σ on f and so either C is fixed by σ which is not possible, or E is fixed by σ. This is not possible too, since otherwise the action of σ on the basis of the fibration would be the identity and so $f = \sigma^*(f)$, a contradiction. Now applying [2, Lemma 5] we find that:

$$2g - 2 = C^2 \leq \frac{2(C \cdot f)^2}{f \cdot \sigma^*(f) + 1} \leq \frac{2 \cdot 9}{3} = 6$$

This implies $g = g(C) \leq 4$. □

Theorem 3.3.2. *Let σ be an automorphism of order* 16 *acting non-symplectically on a $K3$ surface X and assume that $Pic(X) = S(\sigma^8)$ has rank* 6*. Then σ fixes at most one rational curve.*

The corresponding invariants of σ are given in the Table 3.3. In any case $n_{4,13} = n_{5,12} = n_{6,11} = 0$ and we have $(n_{2,15}, n_{3,14}, n_{7,10}, n_{8,9}) = (4, 1, 1, 0)$ in the first case and $(n_{2,15}, n_{3,14}, n_{7,10}, n_{8,9}) = (0, 1, 1, 2)$ in the second case.

m^2	m^1	m	l	r	N	k	N'	$g(C)$	$\mathrm{Pic}(X)$
2	0	0	0	6	6	1	4	7	$U \oplus D_4$
2	0	0	2	4	4	0	2	6	$U(2) \oplus D_4$

Table 3.2: The case rk $\mathrm{Pic}(X) = 6$.

Proof. By the classification theorem for non-symplectic involutions on $K3$ surfaces given by Nikulin in [9, §4] we have that $(g(C), k_{\sigma^8})$ is either equal to $(5,0),(6,1)$ or $(7,2)$. Observe that the case $g(C) = 5$ is not possible. Indeed in this case since $k_{\sigma^8} = 0$ then $k_{\sigma^4} = 0$ too and since C is not fixed by σ^4 by Proposition 3.1.10, we get a contradiction with Proposition 3.1.10 again. Observe that we have $m_\sigma^2 = 2$ so that $m_{\sigma^4} = 8$ by formulas in Remark 3.1.1. This means that the automorphism σ^4 can not have $l_{\sigma^4} > 0$ by [2, Theorem 8.1]. This implies that $l_{\sigma^4} = 0$ and by [2, Theorem 6.1] or [14, Main Theorem 1] we have two possible cases that we recall below, both have $m_\sigma^1 = 0$.

The case $(g(C), k_{\sigma^8}) = (6,1)$. The automorphism σ^4 of order 4 fixes one rational curve and six points on C by [2], [14]. By Riemann-Hurwitz formula applied to the automorphism σ on C we have that:

$$2g(C) - 2 = \deg\ (\sigma_{|C})(2g(D) - 2) + \deg\ R,$$

where $D = C/\langle \sigma_{|C} \rangle$ is the quotient curve and R is the ramification divisor. Since $\deg\ (\sigma_{|C}) = 8$ (where σ acts on C as an automorphism of order eight) we get:

$$2g(C) - 2 = 8(2g(D) - 2) + \deg\ R,$$

thus

$$\frac{2g(C) + 14 - \deg R}{16} = g(D).$$

Such that $\deg\ R = 7N^{'} + 3(2h) + (4\mu)$ since σ acts on C as an automorphism of order 8 , where we denoted by $N^{'}$ the number of fixed points by σ on C, $2h$ the number of interchanged points by σ on it and by 4μ the number of permuted points on C of σ. Since $g(D) \in \mathbb{Z}_{\geq 0}$ we have:

$$2g(C) + 14 - (7N^{'} + 6h + 4\mu) \equiv 0 \mod 16, \tag{I}$$

Hence for $g(C) = 6$ one can find by (I) that either σ exchanges two fixed points and permutes the other four or σ fixes two points and the other four are exchanged two by two. The first case is not possible since then $N = 2$ and by Proposition 3.1.8 we know that $N \geq 4$. So we are in the second case. Since again $N \geq 4$ then the rational curve is invariant but not fixed and so $N = 4$ and by Remark 3.1.9 we have $(n_{3,14}, n_{7,10}, n_{8,9}) = (1,1,2)$ the others $n_{t,s}$ are zero. We have moreover that $k_{\sigma^2} = 1$ and $N_{\sigma^2} = 6$ so combining the Lefschetz formulas we have $r + l + 2m = 6$, $4 = 2 + r - l$, $6 = 2 + r + l - 2m - 2$. That gives $m = 0$ and $r = 4$, $l = 2$. This is the second case in the Table 3.3.

The case $(g(C), k_{\sigma^8}) = (7,2)$. The automorphism σ^4 of order 4 fixes one rational curve, four points on C and two points on the other rational curve see [2], [14]. By relation (I) of Riemann-Hurwitz formula applied to the automorphism σ on C we find that either σ exchanges two by two the four points or it fixes each of the four points. In the first case since $N \geq 4$ we have that the two rational curves are invariant and they contain 2 fixed points each, so that $N = 4$ by Remark 3.1.9. Then $(n_{3,14}, n_{7,10}, n_{8,9}) = (1,1,2)$ so that $k_{\sigma^2} = 1$ and $N_{\sigma^2} = 6$. We have $n_{2,7} + n_{3,6} = 6$ and since $n_{4,5} = 0$ (we have $k_{\sigma^4} = 1$) we get $n_{3,6} = 1$ and $n_{2,7} = 5$. Using Proposition 3.1.8 and 3.1.11 we compute here that $(r, l, m) = (4,2,2)$ and we have $\mathrm{Pic}(X) = U \oplus D_4$ by [2, Theorem 6.1]. By applying Proposition 3.3.1 we know that the $K3$ surface X carries a σ-invariant elliptic fibration with a singular fiber I_0^*. Since the action is not trivial on $\mathrm{Pic}(X)$ the automorphism σ should act non trivially on I_0^*. Since C intersects in three points the fiber I_0^* then the only possibility is that σ exchanges two components of multiplicity one. Then the third point on C would be fixed but this is not possible. So the action of σ on C fixes the four points. Observe that then the number of fixed points for σ^2 satisfies $n_{2,7} + n_{3,6} \geq 4$ so that $k_{\sigma^2} = 1$ by Proposition 3.1.11. This again gives $n_{2,7} + n_{3,6} = 6$ and so $n_{4,5} = 0$ and $n_{2,7} = 5, n_{3,6} = 1$. Finally observe that the case $(N, k) = (8, 0)$ is not possible for σ by Remark 3.1.9 and so we have $(N, k) = (6, 1)$. Again by Remark 3.1.9 we have $(n_{2,15}, n_{3,14}, n_{7,10}) = (4, 1, 1)$. In this case we have $r + l + 2m = 6$, $r - l = 6$, $r + l - 2m = 6$. We find $m = 0$, $r = 6$, $l = 0$. So σ acts trivially on Pic(X) and this is the first case in the table. □

3.4 The rank fourteen case.

We assume finally in this section that $S(\sigma^8) = \text{Pic}(X)$ has rank 14. We recall the notation that will be used here let C denotes the σ^8-fixed curve of genus > 1 (where the case with $g(C) = 1$ and rk Pic$(X) = 14$ is studied in Section 3.2). and N' denotes the number of fixed points that are contained in C. Finally, let k_{σ^j} be the number of fixed rational curves, and N_{σ^j} for $j = 1, 2, 4, 8$ be the number of fixed points in Fix(σ^j), and by $2a_{\sigma^4}$ the number of exchanged smooth rational curves by σ^4 and fixed by σ^8.

Theorem 3.4.1. *Let σ be an automorphism of order 16 acting non symplectically on a $K3$ surface X and assume that $S(\sigma^8) = \text{Pic}(X)$ has rank 14. Then the $K3$ surface is one of the surfaces described in Proposition 3.2.2 with a fixed elliptic curve for the automorphism σ^4 or it has:*

m^2	m^1	m	l	r	N	k	N'	$g(C)$	Pic(X)
1	0	0	1	13	12	1	2	3	$U \oplus D_4 \oplus E_8$
1	0	1	1	11	10	1	2	2	$U(2) \oplus D_4 \oplus E_8$
1	0	1	5	7	4	0	2	2	$U(2) \oplus D_4 \oplus E_8$

Table 3.3: The case rk Pic$(X) = 14$.

Proof. By [9, §4] we know that for the genus $g := g(C)$ of the fixed curve by σ^8 and the number k_{σ^8} of rational curves (different from C) holds:

$$(g, k_{\sigma^8}) = (0, 3),\ (1, 4),\ (2, 5),\ (3, 6)$$

The case $g(C) = 0$. We are in the case of [2, Theorem 5.1] for σ^4, so we have $(r_{\sigma^4}, l_{\sigma^4}, m_{\sigma^4}) = (10, 4, 4)$ with $(N_{\sigma^4}, k_{\sigma^4}, a_{\sigma^4}) = (6, 1, 0)$ since $l_{\sigma^4}, m_{\sigma^4} \in 4\mathbb{Z}$ by Remark 3.1.1. On the other hand, since $N_{\sigma^4} = 6$ and $k_{\sigma^8} = 3$, we have $N = 4, 6, 8$ by Proposition 3.1.8. Moreover since $k_{\sigma^4} = 1$ then k_{σ^2} and k are 0 or 1.

Assume first $k_{\sigma^2} = 0$ since σ^4 acts in a different way on the four rational curves (i.e. σ^4 fixes some curves pointwisely while it leaves the other curves invariant with two fixed points by σ^4 on each one of them), these must be preserved by σ and so also by σ^2. We have $n_{4,5} = 2$, $n_{2,7} = 3 = n_{3,6}$ by Remark 3.1.12. These contradicts Proposition 3.1.11. If $k_{\sigma^2} = 1$ then $n_{4,5} = 0$ and $n_{2,7} = 3 = n_{3,6}$. This again contradicts Proposition 3.1.11.

The case $g(C) = 1$. We have already study this case in Section 3.2, where we proved in Theorem 3.2.2 that σ acts as an automorphism of order four on C and we given the corresponding invariants of σ in this case.

The case $g(C) = 2$. By Proposition 3.1.10 we have $k_{\sigma^4} \geq 1$ so that σ^4 fixes at least a rational curve. Moreover by formulas of Remark 3.1.1 we have $r_{\sigma^4} + l_{\sigma^4} = 14$ and $l_{\sigma^4}, m_{\sigma^4} \in 4\mathbb{Z}$. Observe that $m_{\sigma^4} = 4m_\sigma^2 = 4$. By [2, Theorem 8.1] if $l_{\sigma^4} > 0$ then we have $l_{\sigma^4} + m_{\sigma^4} = 4$ or 8. The first case is not possible, if $l_{\sigma^4} + m_{\sigma^4} = 8$ then $l_{\sigma^4} = 4$ and by [2, Theorem 8.1] we have $k_{\sigma^4} = 1$.

Observe that σ preserves or permutes or exchanges two by two the four rational curves not fixed by σ^4 (where $k_{\sigma^8} = 5$) so that in any case $N_{\sigma^4} \geq 8$. By [2, Proposition 1] we have $N_{\sigma^4} = 6$ which contradicts the previous inequality. Hence $l_{\sigma^4} = 0$ and so σ^4 acts trivially on Pic(X). By [2, Theorem 6.1] we have $(m_{\sigma^4}, r_{\sigma^4}, N_1, N'_{\sigma^4}, k_{\sigma^4}) = (4, 14, 4, 6, 3)$ where $N_{\sigma^4} = N_1 + N'_{\sigma^4}$ and N'_{σ^4} is the number of fixed points on C. So we have 4 points contained in the two rational curves that are σ^4-invariant but not fixed. We call these curves R_1 and R_2. We study now the action of σ and σ^2 on the 5 rational curves, fixed by σ^8, and on C.

<u>The automorphism σ^2</u>. We have $k_{\sigma^2} \leq 3$ and at least one of the five curves is preserved or fixed. By using Remark 3.1.12 we have: $n_{4,5} \in 2\mathbb{Z}$ (points of this type can occur only on the rational curves) and $n_{2,7} + n_{3,6} \leq 10$ (we have $N_{\sigma^4} = 10$, at most 6 fixed points are on C and points of this type are not contained in rational curves that are fixed for σ^4 but can be contained in the two rational curves that are only σ^4-invariant). By using Proposition 3.1.11 we obtain that $k_{\sigma^2} \leq 2$. If $k_{\sigma^2} = 0$, since the action of σ^4 is not the same, then all the rational curves are preserved by σ^2 in particular $n_{4,5} = 6$ and $n_{2,7} \geq 2$ $n_{3,6} \geq 2$. This contradicts Proposition 3.1.11. We are left with the cases with $k_{\sigma^2} = 1$ or $k_{\sigma^2} = 2$.

i) $k_{\sigma^2} = 2$. By Proposition 3.1.11 we get $n_{2,7} + n_{3,6} = 10$ this means that the curve C must contain six fixed points for σ^2 and the other four fixed points are contained in the two σ^4-invariant curves R_1 and R_2. In particular we have $n_{2,7} \geq 2$ and $n_{3,6} \geq 2$, and $n_{4,5} = 2$. Since by Proposition 3.1.11 we have $n_{4,5} = 2n_{3,6} - 4$ we get $n_{3,6} = 3$, $n_{2,7} = 7$, $N_{\sigma^2} = 12$.

ii) $k_{\sigma^2} = 1$. By Proposition 3.1.11 we have $n_{2,7} + n_{3,6} = 6$. Observe that for the same reason as above the remaining rational curves can not be exchanged two by two. So these are invariant. This gives $n_{2,7} \geq 2$, $n_{3,6} \geq 2$ and $n_{4,5} = 4$. Using Proposition 3.1.11 we obtain that $n_{2,7} = n_{3,6} = 3$. And two fixed points are contained in C. The other points on C fixed by σ^4 form a σ-orbit of length four.

<u>The automorphism σ</u>. First observe that using Riemann-Hurwitz formula on C (see relation (I) in Theorem 3.3.2) we have two possibilities: C contains 2 fixed points and the other four points are permuted by σ in one orbit (this is case ii)) or the six points are exchanged two by two and so fixed by σ^2 (this is case i)).

i) In this case σ exchanges two by two the points on C. We have $n_{5,12} = n_{4,13} = 1$ since these two points correspond to the two fixed points with local action $(4, 5)$ for σ^2 and are contained in a rational curve (see Remark 3.1.7). Assume that R_1 and R_2 are not exchanged. We have $n_{2,15} + n_{7,10} + n_{3,14} + n_{6,11} = 4$ and $n_{2,15} = n_{3,14}$, $n_{7,10} = n_{6,11}$. But this contradicts equation (3.1.5) in Proposition 3.1.8. If R_1 and R_2 are exchanged we have $n_{3,14} = n_{6,11} = 0$, $n_{2,15} = n_{7,10} = 0$ and $n_{5,12} = n_{4,13} = 1$. But this contradicts the equality $n_{3,14} - n_{6,11} = 1$ in Proposition 3.1.8.

ii) In this case C contains two fixed points for σ. We have $n_{8,9} = 2w$, with $w = 0, 1$. Moreover by Remark 3.1.7 we have $n_{5,12} = n_{4,13} = 2$ or $n_{5,12} = n_{4,13} = 0$. If $n_{8,9} = 2$

so that $k_\sigma = 0$ an easy computation using the equations of Proposition 3.1.8 shows that the first case with $n_{5,12} = n_{4,13} = 2$ is not possible. If $n_{5,12} = n_{4,13} = 0$ again using Proposition 3.1.8 we find that $n_{3,14} = n_{7,10} = 1$ the other $n_{t,s}$ are zero. One computes $(r_\sigma, l_\sigma, m_\sigma) = (7, 5, 1)$ and we have $\mathrm{Pic}(X) = U(2) \oplus D_4 \oplus E_8$. Observe that in this case the remaining σ^8-fixed rational curves are exchanged two by two by σ. If $n_{8,9} = 0$ so that $k_\sigma = 1$ again one computes using Proposition 3.1.8 that :

$$(N, k, n_{8,9}, n_{2,15}, n_{3,14}, n_{4,13}, n_{5,12}, n_{6,11}, n_{7,10}) = (10, 1, 0, 3, 2, 2, 2, 1, 0)$$

and $(r_\sigma, l_\sigma, m_\sigma) = (11, 1, 1)$. Moreover we have $\mathrm{Pic}(X) = U(2) \oplus D_4 \oplus E_8$.

The case $g(C) = 3$. By Proposition 3.1.10 we have $k_{\sigma^4} \geq 1$ so that σ^4 fixes at least a rational curve. We have moreover by formulas of Remark 3.1.1 that $r_{\sigma^4} + l_{\sigma^4} = 14$ and $l_{\sigma^4}, m_{\sigma^4} \in 4\mathbb{Z}$ and observe that $m_{\sigma^4} = 4m_\sigma^2 = 4$. By [2, Theorem 8.1] if $l_{\sigma^4} > 0$ then we have $l_{\sigma^4} + m_{\sigma^4} = 4$ or 8. The first case is not possible, if $l_{\sigma^4} + m_{\sigma^4} = 8$ then $l_{\sigma^4} = 4$ and by [2, Theorem 8.1] we have $k_{\sigma^4} = 1$. Observe that σ preserves or permutes some of the five rational curve not fixed by σ^4 so that in any case $N_{\sigma^4} \geq 10$. By [2, Proposition 1] we have $N_{\sigma^4} = 6$, which is not possible. Hence $l_{\sigma^4} = 0$ and so σ^4 acts trivially on $\mathrm{Pic}(X)$. By [2, Theorem 6.1] we have $(m_{\sigma^4}, r_{\sigma^4}, n_1, n_2, k_{\sigma^4}) = (4, 14, 6, 4, 3)$ where $N_{\sigma^4} = n_1 + n_2$ and n_2 is the number of fixed points on C. We have hence 6 points contained in the three rational curves that are σ^4-invariant but not fixed. We call these curves T_i, $i = 1, 2, 3$. We study now the action of σ and σ^2 on the 6 rational curves fixed by σ^8 and on C.

<u>The automorphism σ^2</u>. We have $k_{\sigma^2} \leq 3$ and observe that since σ can not permute the four curves, since the action of σ^4 is different, then each curve is preserved by σ^2. Moreover we have $n_{4,5} \in 2\mathbb{Z}$, and these are at most 6, in fact points of this type can occur only on the rational curves, and $n_{2,7} + n_{3,6} \leq 10$ (we have at most 4 fixed points on C and points of this type are not contained in rational curves that are fixed for σ^4, but can be contained in the three rational curves that are only σ^4-invariant). Again by using Proposition 3.1.11 we find that $k_{\sigma^2} \leq 2$. If $k_{\sigma^2} = 0$ then $n_{2,7} + n_{3,6} = 2$ but since all the rational curves are preserved $n_{4,5} = 6$ and we get a contradiction using Proposition 3.1.11. We are left with the cases with $k_{\sigma^2} = 1$ or $k_{\sigma^2} = 2$.

i) $k_{\sigma^2} = 2$. Here we get $n_{2,7} + n_{3,6} = 10$ this means that the curve C must contain four fixed points for σ^2 and the other six points are contained in the three σ^4-invariant curves T_1, T_2 and T_3. In particular we have $n_{2,7} \geq 3$ and $n_{3,6} \geq 3$, $n_{4,5} = 2$. Moreover $n_{4,5} = 2n_{3,6} - 4$ so we get $n_{3,6} = 3$, $n_{2,7} = 7$, $N_{\sigma^2} = 12$ (by Proposition 3.1.11).

ii) $k_{\sigma^2} = 1$: Here we get $n_{2,7} + n_{3,6} = 6$ by Proposition 3.1.11. Observe that for the same reason as above the remaining rational curves can not be exchanged two by two. So these are invariant. This gives $n_{2,7} \geq 3$, $n_{3,6} \geq 3$ and $n_{4,5} = 4$. We get using Proposition 3.1.11 that $n_{2,7} = n_{3,6} = 3$, and so the four points on C fixed by σ^4 form a σ-orbit of length four.

<u>The automorphism σ</u>. By using Riemann-Hurwitz formula there are two possible actions on C: The automorphism σ exchanges 2 points and fixes the other two (this is case i)) or the four points form a σ-orbit (this is case ii)).

i) We have $n_{8,9} = 2w$ and since $k_{\sigma^2} = 2$ we have $0 \leq w \leq 2$. Moreover $n_{5,12} = n_{4,13} = 1$ (since these two points correspond to the two fixed points with local action $(4, 5)$ for σ^2). If $w = 0$ and $k = 0$, so that the two σ^2-fixed curves are exchanged by σ, then using Proposition 3.1.8 one sees that this case is not possible. If $w = 0$ and $k = 2$ using Proposition 3.1.8 we get $N = 14$ which is impossible by looking at the geometry (indeed in this case we have $N \leq 12$).

If $w = 1$, then $k = 1$ and we find $N = 12$ with

$$(N, k, n_{8,9}, n_{2,15}, n_{3,14}, n_{4,13}, n_{5,12}, n_{6,11}, n_{7,10}) = (12, 1, 2, 3, 2, 1, 1, 1, 2).$$

This is the case in the statement.

If $w = 2$ and $k = 0$ this is not possible by using equation in Proposition 3.1.8.

ii) We have $n_{8,9} = 2w$ and since $k_{\sigma^2} = 1$ we have $w = 0, 1$. If $w = 0$ then $k = 1$ and $n_{5,12} = n_{4,13} = 2$ or $n_{5,12} = n_{4,13} = 0$. If $n_{5,12} = n_{4,13} = 2$ we obtain $n_{6,11} = 1$ and $n_{7,10} = 0$ which is impossible since the fixed points by σ are contained in the rational curves that are fixed by σ^8 (see Remark 3.1.7). If $n_{5,12} = n_{4,13} = 0$ then two of the σ^4-fixed curves are exchanged. By using Proposition 3.1.8 we get $n_{7,10} = 1$, $n_{2,15} = 4$, $n_{3,14} = 1$ (the other $n_{t,s}$ are zero), but this is not possible since the isolated points fixed by σ are contained in rational curves (see Remark 3.1.7).

If $w = 1$ then $k = 0$ then again $n_{5,12} = n_{4,13} = 2$ or $n_{5,12} = n_{4,13} = 0$. By using Proposition 3.1.8 we see that the first case is not possible. If $n_{5,12} = n_{4,13} = 0$ then two of the σ^4-fixed curves are exchanged. By Proposition 3.1.8 we find $N = 4$. This is not possible indeed if the curves T_i are preserved then $N = 6$, if two of them are exchanged we get $N = 2$. In any case we get a contradiction. □

Remark 3.4.2. *If* $\operatorname{rk} S(\sigma^8) = 14$ *then the automorphism* σ *acts on* $S(\sigma^8)^\perp \otimes \mathbb{C}$ *by the eight primitive roots of unity* ζ_{16}^i, $i = 1, 3, \ldots, 15$. *In particular each eigenspace is one-dimensional, so by applying the construction for the moduli space of K3 surfaces with non-symplectic automorphisms as described in [25, §11], we see that in fact this is zero dimensional. This is the case in Theorem 3.2.2 and in Theorem 3.4.1. If* $\operatorname{rk} S(\sigma^8) = 6$ *using the same construction as above one finds that the dimension of the moduli space is one.*

3.5 Examples.

In this section we give an example for each case in the classification of the non-symplectic automorphisms of order 16. This shows more precisely that all the cases in the classifiction do exist. We construct all this examples by using elliptic fibrations on $K3$ surfaces. The main definitions and properties of elliptic fibrations, that we need, are contained in the Section 1.3.

Example 3.5.1. Consider the elliptic fibration in Weierstrass form given by :

$$y^2 = x^3 + ax + bt^8$$

where $a, b \in \mathbb{C}$ and the automorphism $\sigma(x, y, t) = (-x, iy, \zeta_{16}^{13}t)$ (recall that $i = \zeta_{16}^4$). By making the coordinate transformation that replac e x by $\lambda^4 x$ and y by $\lambda^6 y$ for a suitable $\lambda \in \mathbb{C}$ we can assume that $a = 1$. Moreover since $b \neq 0$ we can apply a coordinate tranformation to t and so assume that $b = 1$ too. Our equation becomes:

$$y^2 = x^3 + x + t^8.$$

The fibers preserved by σ are over $0, \infty$ and the action at infinity is (see [5, §3]):

$$(x/t^4, y/t^6, 1/t) \longmapsto (-ix/t^4, \zeta_{16}^6 y/t^6, \zeta_{16}^{15} 1/t).$$

The discriminant of the fibration is

$$\Delta(t) = 4 + 27t^{16}.$$

We have that $t = \infty$ is an order eight zero of $\Delta(t)$, and $\Delta(t)$ has 16 simple zeros. Looking in the classification of singular fibers of elliptic fibrations on surfaces (e.g. [17, Section 3]) we see that the fiber over $t = \infty$ is of type IV^* and the fibration has 16 fibers of type I_1. In particular the fiber over $t = 0$ is smooth. By [5, §3] a holomorphic two form is given by $(dt \wedge dx)/2y$ and so the action of σ on it is by multiplication by ζ_{16}. In fact we can understand the local action of the automorphism σ at the fixed points on C. If we look at the elliptic fibration locally around the fiber over $t = 0$ the equation in $\mathbb{P}^2 \times \mathbb{C}$ is given by:

$$G(x, y, z, t) := zy^2 - (x^3 + z^2x + z^3t^8) = 0$$

where $(x : y : z)$ are the homogeneous coordinates of $\mathbb{P}^2$ and the two fixed points for the automorphism σ on the fiber $t = 0$ are $p_0 := (0 : 1 : 0)$ and $p_1 := (0 : 0 : 1)$. In the chart $z = 1$ and on the open subset $\partial G(x, y, 1, 0)/\partial x \neq 0$ that contains the fixed point $p_1 = (0 : 0 : 1)$, a one form for the elliptic curve over $t = 0$ is:

$$dy/(\partial G(x, y, 1, 0)/\partial x) = dy/(-3x^2 - 1).$$

Here the action of σ is a multiplication by i so that the action on the holomorphic two form

$$dt \wedge (dy/(-3x^2 - 1))$$

is a multiplication by ζ_{16} as expected, and we see that the local action is of type $P^{4,13}$. Doing a similar computation in an open subset of the chart $y = 1$ that contains the fixed point p_0 we find again the same local action. So we are in the first case of the Theorem 3.2.2 with $N = 8$. On the other hand the fibration admits also the automorphism $\gamma(x, y, t) = (-x, -iy, \zeta_{16}^5 t)$. This acts also by multiplication by ζ_{16} on the holomorphic two form, so γ is not a power of σ. In this case a similar computation as above shows that the local action at the fixed points on the fiber C is of type $P^{5,12}$, so we are in the second case of the Theorem 3.2.2.

Example 3.5.2. 1) The case $g(C) = 7$, $(r, l) = (6, 0)$, $\mathrm{Pic}(X) = U \oplus D_4$.

Consider as in [7, Section 3.4] the elliptic fibration:

$$y^2 = x^3 + t^2x + (bt^3 + t^{11})$$

with $b \in \mathbb{C}$ and with the automorphism $\sigma(x, y, t) = (\zeta_{16}^2 x, \zeta_{16}^3 y, \zeta_{16}^2 t)$ (we write here the fibration in a slightly different way as given in [7]). On $t = 0$ the fibration has a fiber of type I_0^* and on $t = \infty$ the fibration has a fiber of type II. The action on the holomorphic two form $(dx \wedge dt)/2y$ is a multiplication by ζ_{16}. This is a one dimensional family and for generic b the action is trivial on $\mathrm{Pic}(X)$. So we are in the first case of Theorem 3.3.2. Observe that the fiber of type I_0^* contains the four fixed points with local action of type $P^{2,15}$ and the invariant elliptic cuspidal curve over $t = \infty$ contains the fixed point with local action $P^{14,3}$ (which is also contained on the section of the fibration) and the point of type $P^{7,10}$. In particular observe that the curve C of genus 7 meets the fiber of type II at the singular point with multiplicity 3.

Observe that if $b = 0$ we get the elliptic fibration with the order 32 automorphism

$$\sigma_{32}(x, y, t) = (\zeta_{32}^{18} x, \zeta_{32}^{11} y, \zeta_{32}^2 t)$$

as described e.g. in [15]. The automorphism σ is the square of the automorphism σ_{32}^{25}.

2) The case $g(C)=6$, $(r,l)=(4,2)$, $\mathrm{Pic}(X)=U(2)\oplus D_4$.

The surfaces of this kind are described in the paper [13] and they are double covers of $\mathbb{P}^2$ ramified on a reducible sextic which is the product of a smooth quintic and a line. We consider the special family with equation in $\mathbb{P}(3,1,1,1)$:

$$z^2=x_0(\alpha_0x_0^4x_2+\beta_0x_1^5+\beta_1x_1^3x_2^2+\beta_2x_1x_2^4).$$

Observe that the quintic curve is smooth and the K3 surface has five A_1 singularities over the points of intersection of the quintic curve and the line. The K3 surface carries the order 16 non-symplectic automorphism

$$\sigma(z:x_0:x_1:x_2)\mapsto(\zeta_{16}^3z:x_0:\zeta_8^7x_1:\zeta_8^3x_2).$$

This acts by multiplication by ζ_{16} on the holomorphic two form:

$$(dx\wedge dy)/\sqrt{f}$$

where $f(x,y)=0$ is the equation of the ramification sextic in the local coordinates x and y. An easy computation shows that the automorphism fixes the points:

$$(0:1:0:0),\qquad (0:0:1:0),\qquad (0:0:0:1)$$

Observe that the point $(0:0:0:1)$ is in fact one of the five A_1 singularities on the K3 surface. If we resolve it we find a fixed point on the strict transform of C which is the quintic curve on $\mathbb{P}^2$ (that have genus six) and one fixed point on the strict transform of L which denotes the curve $\{x_0=0\}$. The other two fixed points are contained respectively in C and L (and their respective strict transforms). Observe that the automorphism σ exchanges two by two the other points of intersection of C with L.

Example 3.5.3. 1) **The case** $g(C)=3$ (see [14]). Consider the elliptic fibration:

$$y^2=x^3+t^2x+t^7$$

This carries the order 16 automorphism $\sigma(x,y,t)=(\zeta_{16}^2x,\zeta_{16}^{11}y,\zeta_{16}^{10}t)$. The discriminant is $t^6(4+27t^8)$ so over $t=0$ the fibration has a fiber of type I_0^* and over $t=\infty$ the fibration has a fiber of type II^*. The automorphism σ preserves the II^* fiber and fixes the component of multiplicity 6. The genus 3 curve cuts the fiber II^* in the component of multiplicity 3. Finally σ exchanges two curves in the I_0^* fiber (this corresponds to $l_\sigma=1$), it leaves invariant the component of multiplicity two and contains two fixed point on it. Using Remark 3.1.7 it is easy to find the local action at the 12 fixed points. In this case we have $\mathrm{Pic}(X)=U\oplus D_4\oplus E_8$.

2) **The case** $g(C)=2$ **and** $k_\sigma=0$. We consider the K3 surface double cover of $\mathbb{P}^2$ ramified on a special reducible sextic as in Example 3.5.2, 2). We consider the quintic with a special equation, more precisely we assume that the reducible sextic $(L=\{x_0=0\})\cup C$ has the equation:

$$x_0(x_0^4x_2+x_1^5-2x_1^3x_2^2+x_2^4x_1)=0,$$

and recall that the automorphism is:

$$\sigma:(z:x_0:x_1:x_2)\mapsto(\zeta_{16}^3z:x_0:\zeta_8^7x_1:\zeta_8^3x_2).$$

The line $L = \{x_0 = 0\}$ meets the quintic in the point $(0 : 0 : 1)$ and two further points $(0 : 1 : 1)$ and $(0 : -1 : 1)$, that are in fact exchanged by the automorphism σ. By studying the partial derivatives of the equation of C one sees that these are singular points. These are in fact A_3 singularities. We explain the computations in detail for the point $(0 : 1 : 1)$. In the chart $x_2 = 1$ the equation of C becomes:

$$x_0^4 + x_1^5 - 2x_1^3 + x_1 = 0$$

We translate the point $(0, 1)$ to the origin and we get an equation in new local coordinates (here $x_0 = y$):

$$x^2(x^3 + 5x^2 + 8x + 4) + y^4 = 0$$

So we have a double point at $(0, 0)$ and by making a coordinates transformation as in [22, Ch. II, section 8] we obtain the local equation:

$$x^2 + y^4 = 0$$

which is an A_3 singularity. Now as explained again in [22, Ch. II, section 8] or also in [13, Lemma 3.15] this gives a D_6 singularity of the reducible ramification sextic. The same happens at the point $(0 : -1 : 1)$ since the two points are exchanged by σ. This means that the K3 surface defined by

$$z^2 = x_0(x_0^4x_2 + x_1^5 - 2x_1^3x_2^2 + x_2^4x_1)$$

has two D_6 singularities and one A_1 singularity (coming from the intersection point $(0 : 0 : 1)$). Let X be the minimal desingularization of the double cover. The rank of the Picard group is at least 14 but since the automorphism of order 16 acts non-symplectically on it, the rank is exactly 14 and $\mathrm{Pic}(X) = U(2) \oplus D_4 \oplus E_8$. Observe that the (-2)-curve coming from the resolution of the A_1 singularity can not be fixed, because it intersects C and L on X (we call again in this way the strict transforms) that are σ^8-fixed. Moreover since the two D_6 singularities are exchanged we have $k = 0$. Observe that the induced automorphism on $\mathbb{P}^2$ fixes also the point $(0 : 1 : 0) \in L$ and the point $(1 : 0 : 0) \in C$ which together with the two intersection points with L and C of the exceptional (-2)-curve on the A_1 singularity gives $N = 4$.

3) **The case** $g(C) = 2$ **and** $k_\sigma = 1$ (see [26]). We consider the elliptic fibration in Weierstrass form with the non-symplectic automorphism of order 16 :

$$y^2 = x^3 + t^3(t^4 - 1)x, \qquad \sigma : (x, y, t) \mapsto (\zeta_{16}^6 x, \zeta_{16}^9 y, \zeta_{16}^4 t)$$

This fibration has five fibers III (one over $t = \infty$) and one fiber III^* over $t = 0$. An easy computation using the local action at the fixed points shows that we have $k_\sigma = 1$ and 10 isolated fixed points.

Appendix A

X defined by a quartic.

In this appendix we classify all quartic surfaces that are (affine) invariant for the action of some automorphism of order 8 of $\mathbb{P}^3$ acting non-symplecticaly on the quartic.

Classification:

Let X be a $K3$ surface defined on $\mathbb{P}^3$ as the zero set of a homogeneous polynomial $f_4 \in \mathbb{C}[X_0, X_1, X_2, X_3]$ of degree four. And let μ be a purely non-symplectic automorphism of order four on it. Then the automorphism μ and the surface X are one of the following cases (we consider here only equations of X that are affine invariants):

[1]

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, X_1, X_2, X_3)$$

or

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, -X_1, -X_2, -X_3).$$

Where

$$\boxed{\begin{array}{c} X = a_1X_0^4 + a_2X_1^4 + a_3X_2^4 + a_4X_3^4 + a_5X_1^3X_2 + a_6X_1X_2^3 + a_7X_1^3X_3 + a_8X_1X_3^3 \\ +a_9X_1^2X_2^2 + a_{10}X_1^2X_3^2 + a_{11}X_2^2X_3^2 + a_{12}X_2^3X_3 + a_{13}X_2X_3^3 + \\ a_{14}X_1^2X_2X_3 + a_{15}X_1X_2^2X_3 + a_{16}X_1X_2X_3^2. \end{array}}$$

[2]

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, -X_1, -X_2, X_3)$$

or

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, X_1, X_2, -X_3).$$

Where

$$\boxed{\begin{array}{c} X = a_1X_0^4 + a_2X_1^4 + a_3X_2^4 + a_4X_3^4 + a_5X_1^3X_2 + a_6X_1X_2^3 + a_7X_1^2X_3^2 + a_8X_1^2X_2^2 \\ +a_9X_2^2X_3^2 + a_{10}X_1X_2X_3^2 + a_{11}X_0^2X_1X_3 + a_{12}X_0^2X_2X_3. \end{array}}$$

[3]

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, iX_1, iX_2, X_3)$$

or

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, iX_1, iX_2, -X_3).$$

Where

$$X = a_1X_0^4 + a_2X_1^4 + a_3X_2^4 + a_4X_3^4 + a_5X_1^3X_2 + a_6X_1X_2^3 + a_7X_0^3X_1 + a_8X_0X_1^3 + a_9X_0^3X_2 + a_{10}X_0X_2^3 + a_{11}X_0^2X_1X_2 + a_{12}X_0X_1^2X_2 + a_{13}X_0X_1X_2^2 + a_{14}X_1^2X_2^2 + a_{15}X_0^2X_1^2 + a_{16}X_0^2X_2^2.$$

[4]

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, iX_1, -iX_2, X_3)$$

or

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, iX_1, -iX_2, -X_3).$$

Where

$$X = a_1X_0^4 + a_2X_1^4 + a_3X_2^4 + a_4X_3^4 + a_5X_0^3X_1 + a_6X_0X_1^3 + a_7X_0X_1X_2^2 + a_8X_0X_2X_3^2 + a_9X_1X_2X_3^2 + a_{10}X_1^2X_2^2 + a_{11}X_0^2X_1^2 + a_{12}X_0^2X_2^2.$$

Now let σ be a purely non-symplectic automorphism of order 8 such that $\sigma^2 = \mu$ where

$$\sigma(X_0, X_1, X_2, X_3) = (\zeta_8^a X_0, \zeta_8^b X_1, \zeta_8^c X_2, \zeta_8^d X_3).$$

Thus we are in one of the following cases:

[1] If μ is either

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, X_1, X_2, X_3)$$

or

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, -X_1, -X_2, -X_3).$$

Then we have :

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^3X_2 + a_5X_1X_2^3 + a_6X_1^3X_3 + a_7X_1X_3^3 + a_8X_1^2X_2^2 + a_9X_1^2X_3^2 + a_{10}X_2^2X_3^2 + a_{11}X_2^3X_3 + a_{12}X_2X_3^3 + a_{13}X_1^2X_2X_3 + a_{14}X_1X_2^2X_3 + a_{15}X_1X_2X_3^2.$$

Where

$$(a, b, c, d) = (1, 4, 4, 4), (1, 8, 8, 8).$$

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^3X_2 + a_5X_1X_2^3 + a_6X_1^2X_2^2 + a_7X_1^2X_3^2 + a_8X_2^2X_3^2 + a_9X_1X_2X_3^2.$$

Where

$$(a, b, c, d) = (1, 4, 4, 8), (1, 8, 8, 4).$$

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^3X_3 + a_5X_1X_3^3 + a_6X_1^2X_2^2 + a_7X_1^2X_3^2 + a_8X_2^2X_3^2 + a_9X_1X_2^2X_3.$$

Where

$$(a, b, c, d) = (1, 4, 8, 4), (1, 8, 4, 4).$$

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^2X_2^2 + a_5X_1^2X_3^2 + a_6X_2^2X_3^2 + a_7X_2^3X_3 + a_8X_2X_3^3 + a_9X_1^2X_2X_3$$

Where

$$(a, b, c, d) = (1, 4, 8, 8), (1, 8, 4, 4).$$

[2] If μ is either

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, -X_1, -X_2, X_3)$$

or

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, X_1, X_2, -X_3).$$

Then we have the following:

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^3X_2 + a_5X_1X_2^3 + a_6X_1^2X_2^2 + a_7X_0^2X_1X_3 + a_8X_0^2X_2X_3.$$

Where

$$(a, b, c, d) = (1, 2, 2, 4)(1, 2, 2, 8)(1, 6, 6, 8)(1, 4, 4, 2)(1, 8, 8, 6).$$

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^2X_2^2 + a_5X_1X_2X_3^2 + a_6X_0^2X_2X_3 .$$

Where

$$(a, b, c, d) = (1, 2, 6, 8)(1, 8, 4, 2).$$

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^2X_2^2 + a_5X_1X_2X_3^2 + a_6X_0^2X_1X_3.$$

Where

$$(a, b, c, d) = (1, 2, 6, 4)(1, 8, 4, 6).$$

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^2X_2^2 + a_5X_1X_2X_3^2 + a_6X_0^2X_1X_3.$$

Where

$$(a, b, c, d) = (1, 6, 2, 8)(1, 4, 8, 2).$$

$$X = a_1X_1^4 + a_2X_2^4 + a_3X_3^4 + a_4X_1^3X_2 + a_5X_1X_2^3 + a_6X_1^2X_2^2.$$

Where

$$(a, b, c, d) = (1, 4, 4, 6)(1, 8, 8, 2).$$

[3] If μ is either

$$\mu(X_0, X_1, X_2, X_3) = (iX_0, iX_1, iX_2, X_3)$$

or

$$\mu(X_0, X_1, X_2, X_3) \;=\; (iX_0, iX_1, iX_2, -X_3).$$

Then we get:

$$\boxed{X = a_1X_3^4.}$$

Where

$$(a,b,c,d) = (1,1,1,2), (1,1,1,6), (1,1,1,4), (1,1,1,8).$$

$$\boxed{X = a_1X_3^4 + a_2X_1^3X_2 + a_3X_1X_2^3 + a_4X_0^3X_2 + a_5X_0X_2^3 + a_6X_0^2X_1X_2 + a_7X_0X_1^2X_2}$$

Where

$$(a,b,c,d) = (1,1,5,2), (1,1,5,6), (1,1,5,8), (1,1,5,4).$$

$$\boxed{X = a_1X_3^4 + a_2X_0^3X_1 + a_3X_0X_1^3 + a_4X_0^3X_2 + a_5X_0X_2^3 + a_6X_0X_1^2X_2 + a_7X_0X_1X_2^2 +.}$$

Where

$$(a,b,c,d) = (1,5,5,2), (1,5,5,6), (1,5,5,4), (1,5,5,8).$$

[4] If μ is either

$$\mu(X_0, X_1, X_2, X_3) \;=\; (iX_0, iX_1, -iX_2, X_3)$$

or

$$\mu(X_0, X_1, X_2, X_3) \;=\; (iX_0, iX_1, -iX_2, -X_3).$$

Then we have :

$$\boxed{X = a_1X_3^4 + a_2X_0X_1X_2^2 + a_3X_0X_2X_3^2 + a_4X_1X_2X_3^2 + a_5X_1^2X_2^2 + a_6X_0^2X_2^2.}$$

Where

$$(a,b,c,d) = (1,1,3,2), (1,1,3,6), (1,1,7,4), (1,1,7,8).$$

$$\boxed{X = a_1X_3^4 + a_2X_0^3X_1 + a_3X_0X_1^3 + a_4X_0X_2X_3^2 + a_5X_1^2X_2^2 + a_6X_0^2X_2^2.}$$

Where

$$(a,b,c,d) = (1,5,3,2), (1,5,3,6), (1,5,7,4), (1,5,7,8).$$

$$\boxed{X = a_1X_3^4 + a_2X_0^3X_1 + a_3X_0X_1^3 + a_4X_1X_2X_3^2 + a_5X_1^2X_2^2 + a_6X_0^2X_2^2.}$$

Where

$$(a,b,c,d) = (1,5,7,2), (1,5,7,6), (1,5,3,4), (1,5,3,8).$$

$$X = a_1X_3^4 + a_2X_0X_1X_2^2 + a_3X_1^2X_2^2 + a_4X_0^2X_2^2.$$

Where

$$(a, b, c, d) = (1, 1, 7, 2), (1, 1, 7, 6), (1, 1, 3, 4), (1, 1, 3, 8).$$

Note that the surface X in all these previous cases is not smooth and so the proof of Remark 2.5.4 in §2.5.1 holds.

Appendix B

The Case $l = 0$.

In this appendix we study the case $l = 0$, so that $r_{\sigma^2} = r$ (i.e. σ^* acts as the identity on $S(\sigma^2)$). We classify more precisely non-symplectic automorphisms of order eight on $K3$ surfaces which act trivially on the Picard lattice and we give an independent proof (not based on the classification of order four automorphisms [2]) of [14, Proposition 5.5]. We recall first some notations. Let $2h$ be the number of interchanged points by σ on the curve $C \subset \mathrm{Fix}(\sigma^4)$ with genus $g(C) > 0$, and N' respectively N'_{σ^2} be the number of fixed points by σ respectively σ^2 contained in the curve C. And we denote by k respectively k_{σ^2} the number of smooth rational curves fixed by σ respectively σ^2, and by N respectively N_{σ^2} the number of isolated points in $\mathrm{Fix}(\sigma)$ respectively $\mathrm{Fix}(\sigma^2)$. Finally, we denote by $2a$ the number of exchanged smooth rational curves by σ and fixed (that means pointwisely fixed) by σ^2, $2A$ the number of smooth rational curves interchanged by σ and invariants (but not pointwise fixed) by σ^2, and by $2a_{\sigma^2}$ the number of smooth rational curves interchanged by σ^2 and fixed by σ^4.

Proposition B.0.4. *Let σ be a purely non-symplectic automorphism of order 8 on a $K3$ surface X such that $l = 0$. Then $a = A = 0$ and $n_4 = r - 6k$.*

Proof. Assume that A is not zero, thus there are two different smooth rational curves G_1, G'_1 invariants by σ^2 such that $\sigma(G_1) = G'_1$ and $\sigma(G'_1) = G_1$. Let $g = [G_1] - [G'_1]$ in Pic(X). Thus $\sigma^*(g) = -g$ such that $g \neq 0$ which contradicts with $l = 0$ (where l is the rank of the eigenspace of σ^* in $H^2(X, \mathbb{Z})$ with eigenvalues -1). By the same argument we can prove that $a = 0$ when $l = 0$. On other hand, by Proposition 1 we have that $n_2 + n_3 + n_4 = 2 + r - 2k$ since $l = 0$, thus $n_4 = r - 6k$ (where $n_2 + n_3 = 4k + 2$). □

We give now a useful proposition showing that $N_{\sigma^2} = n_2 + n_3$ if $l = 0$. This proposition proves [14, Lemma 5.2] in a more general case.

Proposition B.0.5. *If X has a non-symplectic automorphism σ of order 8 such that $l = 0$, then for any isolated point $p \in \mathrm{Fix}(\sigma^2)$ we have $\sigma(p) = p$.*

Proof. Observe that if σ^2 fixes a curve C of genus $g(C) > 0$, then all isolated points by σ on it are of type $P^{4,5}$. Thus $\sigma(p) = p$ for all $p \in \mathrm{Fix}(\sigma^2)$ since $A = 0$ by Proposition B.0.4 (where $N_{\sigma^2} = n_2 + n_3 + 4A$). Otherwise, all fixed curves by σ^2 are rational. Thus by computing the topological Lefschetz formula since $A = a = 0$ and $g_{\sigma^2} = 0$ we get:

$$\begin{aligned}\chi(\mathrm{Fix}(\sigma^2)) - \chi(\mathrm{Fix}(\sigma)) &= 2k_{\sigma^2} + N_{\sigma^2} - (2k + N)\\ &= 2(k + n_4/2) + (n_2 + n_3 + 2h) - (2k + n_2 + n_3 + n_4)\\ &= 2h.\end{aligned}$$

On other hand we have that:

$$\begin{aligned}\chi(\mathrm{Fix}(\sigma^2)) - \chi(\mathrm{Fix}(\sigma)) &= 2 + r_{\sigma^2} - l_{\sigma^2} - (2 + r + l)\\ &= -2m.\end{aligned}$$

Hence we get $2h = -2m$, so that $h = m = 0$ and $N_{\sigma^2} = n_2 + n_3$ when $l = 0$ (where $N_{\sigma^2} = (n_2 + n_3) + 4A + 2h$ in this case). □

Remark B.0.6. *As a direct consequence of the previous proposition one gets that: If σ is a purely non-symplectic automorphism of order eight on a $K3$ surface X such that $l = 0$ and all the fixed curves by σ^2 are rational, then $\underline{h = m = 0}$ and the following relation hold.*

$$k_{\sigma^2} = 2k - 1. \tag{B.0.1}$$

In fact one can obtain $m = h = 0$ by computing the difference $\chi(\mathrm{Fix}(\sigma^2)) - \chi(\mathrm{Fix}(\sigma))$ topologically and using the Lefschetz's formula. On the other hand, using the fact that $N_{\sigma^2} = n_2 + n_3 = 4k + 2$ and the relation $N_{\sigma^2} = 2k_{\sigma^2} + 4$ in [2, Proposition 1] we find the relation (B.0.1).
Moreover since $a = 0$ and the fixed points by σ on the curve C of genus $g(C) > 0$ are of type $P^{2,7}, P^{3,6}$ we get:

$$k_{\sigma^2} = k + n_4/2. \tag{B.0.2}$$

Lemma B.0.7. *Let σ be a purely non-symplectic automorphism on a $K3$ surface X such that $l = 0$. Then σ acts trivially on* Pic(X) *if and only if $g(C) = 0$ for all $C \subset \mathrm{Fix}(\sigma^2)$.*

Proof. Observe at the first that if all the fixed curves by σ^2 are rational, then by Proposition B.0.5 and computing the difference $\chi(\mathrm{Fix}(\sigma^2)) - \chi(\mathrm{Fix}(\sigma))$ topologically and using the Lefschetz's formula one gets that $m = h = 0$ (i.e. the automorphism σ acts triviale on Pic(X)). On the other hand, suppose that σ^2 fixes a curve C of genus $g(C) > 0$. By [2, Proposition 1] and computing again $\chi(\mathrm{Fix}(\sigma^2)) - \chi(\mathrm{Fix}(\sigma))$ topologically and using the Lefschetz's formula we get that

$$2m = 2g(C) + N'. \tag{B.0.3}$$

On the other hand, since $n_2 + n_3 = N_{\sigma^2}$ and $k_{\sigma^2} = k + \frac{n_4 - N'}{2}$ by Proposition 2.1.8 and [2, Proposition 1] we have that :

$$n_2 + n_3 = 2 - 2g(C) + 2k + n_4 - N'.$$

Observe that the isolated fixed points by σ of type $P^{2,7}, P^{3,6}$ are contained in smooth rational curves, so that $n_2 = n_3$ (see Remark 2.1.10) and $n_4 = 2k + 2$ by Proposition 2.1.8. We return to the previous relation and we get :

$$(4k + 2) = 2 - 2g(C) + 2k + (2k + 2) - N' + 4, \tag{B.0.4}$$

so that

$$2g(C) + N' = 6. \tag{B.0.5}$$

By (B.0.4) and (B.0.5) we find that $2m = 4$ hence σ does not act trivially on Pic(X). □

Proposition B.0.8. *Let σ be a purely non-symplectic that automorphism of order eight such that $l = 0$. Then if $C \subset \mathrm{Fix}(\sigma^2)$ with $g(C) > 0$ we have C is a smooth elliptic curve. Moreover σ acts on C as an involution with the invariants $(r, m, m_1) = (10, 4, 1)$ and $(k, N) = (1, 10)$.*

Proof. In fact by relation (B.0.5) in Lemma B.0.7 one finds the following possibilities: $(g(C), N^{'}) =$ (1,4), (2,2) and (3,0). Moreover, we get that $m = l_{\sigma^2} = 4$. The last two cases are not possible by [2, Theorem 4.1] (since $l_{\sigma^2} = 4$). On the other hand, σ acts on the elliptic curve C in the first case as an involution with four fixed points of type $P^{4,5}$. More precisely, by Proposition 2.1.8 we have $N = r + 2 - 2k$ so we get $r = 8k + 2$. In fact since the isolated fixed points of type $P^{2,7}, P^{3,6}$ are contained in smooth rational curves we have $n_2 = n_3$ (see Remark 2.1.11), so by Proposition 2.1.8 one finds that $n_4 = 2 + 2k$ and thus $N = (4k + 2) + 2k + 2 = r + 2 - 2k$. We can moreover obtain that $r_{\sigma^4} = 8k + 6$ where $l = 0, m = 4$. Observe that $k \leq 1$ since $r_{\sigma^4} = \mathrm{rk}\ \mathrm{Pic}(X) \leq 20$. On the other hand,$k \geq 1$ since $n_4 \geq 4$ where σ fixes four points of type $P^{4,5}$ on the curve C. So that $k = 1$ and $(r, m_1) = (10, 1)$. □

Theorem B.0.9. *Let σ be a purely non-symplectic automorphisim of order eight on a $K3$ surface X acting trivialy on $Pic(X)$, then $k = 1$ and we are in one of the following cases:*

m_1	r	$N^{'}$	N	(n_2, n_3, n_4)	g_{σ^4}	k	$S(\sigma^4)$
4	6	4	6	$(5, 1, 0)$	7	1	$U \oplus D_4$
4	6	6	6	$(5, 1, 0)$	6	1	$U(2) \oplus D_4$
2	14	4	12	$(7, 3, 2)$	3	2	$U \oplus D_4 \oplus E_8$

Proof. Observe at first that by Lemma B.0.7 all the fixed curves by σ^2 are rational. By Remark B.0.6 we have moreover that $N^{'} = N^{'}_{\sigma^2}$ since $h = 0$ and by relation (B.0.1) we get $k \geq 1$.
If $k = 1$ then $k_{\sigma^2} = 1$ by relation (B.0.1) of Remark B.0.6 and so $n_4 = 0$ by (B.0.2) of the same remark. Thus $(n_2, n_3) = (5, 1)$ and $r = 6$ by Proposition 2.1.8. If $k = 2$ then $n_4 = 2$ since $k_{\sigma^2} = 3 = 2(2) - 1$ by Remark B.0.6 again. So that $(n_2, n_3) = (7, 3)$ and $r = 14$ by Proposition 2.1.8. Similarly one can obtain that for $k = 3$ then $n_4 = 4$ and so $r_{\sigma^4} = r = 22$ by Proposition 2.1.8. That gives a contradiction since $rk\mathrm{Pic(X)} \leq 20$. Hence $r_{\sigma^4} = \mathrm{rk}\ \mathrm{Pic}(X)$ is either 6 or 14 if σ acts as an identity on $\mathrm{Pic}(X)$.

On the other hand, the number of fixed rational curves by σ^4 is given by $k_{\sigma^4} = k_{\sigma^2} + \frac{N_{\sigma^2} - N^{'}_{\sigma^2}}{2} + 2a_{\sigma^2}$. We get by the same argument in the proof of Proposition B.0.4 that $a_{\sigma^2} = 0$, and since $N_{\sigma^2} = n_2 + n_3$ by Proposition B.0.5 and $N^{'} = N^{'}_{\sigma^2}$, $k_{\sigma^2} = 2k - 1$ by Remark B.0.6 one obtains that $k_{\sigma^4} = 2k - 1 + \frac{N - N^{'}}{2}$. Using the classification of non-symplectic involutions given by Nikulin [9, § 4] (see also [4, Figure 1]) where rk $\mathrm{Pic}(X) = 6, 14$ we get the following possibilities $(k_{\sigma^4}, r, N^{'}, g(C))$=(2,6,4,7),(1,6,6,6),(6,14,4,3),(5,14,6,2) and $(4, 14, 8, 1)$. Observe that the last two cases are not possible by Riemann-Hurwitz formula (see (I) in Theorem 2.7.1).

By the classification theorem of 2-elementary even lattices [10, Theorem 3.6.2] we get that $S(\sigma^4) = U \oplus D_4$, $U(2) \oplus D_4$ for $g(C) = 7, 6$ respectively and $S(\sigma^4) = U \oplus D_4 \oplus E_8$ for $g(C) = 3$.

Note that we can use [2, Theorem 6.1] and Riemann-Hurwitz formula with the fact that $N^{'}_{\sigma^2} = N^{'}$ to obtain the cases appearing in the table. But in this proof we have tried to find the invariants of σ without using the order four automorphism. □

Appendix C

Tables.

In this Appendix we give the tables for the complete classification of the non-symplectic automorphisms of order 8 on a K3 surface. These show the invariants of non-symplectic automorphisms of order eight in all the possible cases. Moreover they also show the cases when we have an example, we indicate these with $\surd$, or when there is not an example , we refer to this case with X, (and we give the number of examples if we have more than one). We recall now the invariants that appear in the tables: we denote by k respectively k_{σ^2} the number of smooth rational curves fixed by σ respectively σ^2. And by N the number of isolated points in $\mathrm{Fix}(\sigma)$. Let N' respectively N'_{σ^2} denotes the number of fixed points by σ respectively σ^2 contained in a curve $C \subseteq \mathrm{Fix}(\sigma^4)$ of genus $g \geq 1$, $4s = 2a_{\sigma^2}$ denotes the number of smooth curves that are permuted by σ, interchanged by σ^2 and fixed by σ^4. And we denote by $2a$ the number of exchanged smooth rational curves by σ and fixed (that means pointwisely fixed) by σ^2, $2A$ the number of smooth rational curves interchanged by σ and invariants (but not pointwise fixed) by σ^2, $2h$ the number of interchanged points by σ on the curve C, and by g_{σ^i} for $i = 1, 2, 4$ the genus of the curve $C \subset \mathrm{Fix}(\sigma^{\mathrm{i}})$. Finally, we denote by r, l, m and m_1 the rank of the eigenspaces of $(\sigma)^*$ in $H^2(X, \mathbb{C})$ relative to the eigenvalues $1, -1, i$ and ζ_8 respectively.

m_1	m	r	l	(n_2, n_3, n_4)	N	k	A	type of C'	Example
3	2	3	3	$(2, 0, 0)$	2	0	0	I_0	$\surd$
2	2	6	4	$(1, 1, 2)$	4	0	1	IV^*	$\surd$

Table C.1: The case g=1, $C \subseteq \mathrm{Fix}(\sigma)$.

m_1	m	r	l	N	(n_2, n_3, n_4)	N'	k	a	A	type of C'	Example
3	2	3	3	2	$(2, 0, 0)$	0	0	0	0	I_0	$\surd$
3	2	5	1	6	(0,2,4)	4	0	0	0	I_0	$\surd$
2	2	6	4	4	$(1, 1, 2)$	0	0	0	1	IV^*	X
2	2	10	0	10	$(3, 3, 4)$	4	1	0	0	IV^*	$\surd\surd$

Table C.2: The case g=1 , $\mathrm{Fix}(\sigma^2) \supseteq \mathrm{C} \nsubseteq \mathrm{Fix}(\sigma)$.

m_1	m	r	l	N	(n_2,n_3,n_4)	N'	k	a	type of C'	Example
3	2	3	3	2	(2,0,0)	2	0	0	I_0	X
2	1	10	2	8	$(4,2,2)$	2	1	0	I_8	$\surd$
2	1	8	4	6	(0,2,4)	2	0	0	I_8	$\surd\surd$
2	1	6	6	2	(2,0,0)	2	0	1	I_8	$\surd\surd$
2	3	4	4	2	(2,0,0)	2	0	$0(s=1)$	I_8	X
1	0	17	1	14	(6,4,4)	2	2	0	I_{16}	$\surd\surd\surd$
1	0	11	7	6	(0,2,4)	2	0	1	I_{16}	$\surd\surd$
1	0	9	9	2	(2,0,0)	2	0	2	I_{16}	$\surd$
1	4	5	5	2	(2,0,0)	2	0	$0(s=2)$	I_{16}	X
2	2	6	4	4	(1,1,2)	0	0	0	IV*	X

Table C.3: The case $g=1, \mathrm{Fix}(\sigma^4) \supseteq \mathrm{C} \nsubseteq \mathrm{Fix}(\sigma^2)$.

r	l	m_1	(n_2,n_3,n_4)	N'	h	k	a	g_{σ^4}	$S(\sigma^4)$	Example
1	1	5	(2,0,0)	2	1	0	0	9	$U(2)$	X
6	0	4	(5,1,0)	4	0	1	0	7	$U\oplus D_4$	$\surd$
4	2	4	(1,1,2)	2	2	0	0	6	$U(2)\oplus D_4$	X
6	0	4	(5,1,0)	6	0	1	0			$\surd$
7	3	3	(0,2,4)	2	1	0	0	5	$U(2)\oplus E_8$	X
9	1	3	(2,0,0)	2	1	0	1			X
9	1	3	(4,2,2)	2	1	1	0			$\surd$
9	1	3	(4,2,2)	4	1	1	0	4	$U\oplus D_4^{\oplus 2}$	$\surd$
7	3	3	(0,2,4)	2	3	0	0	3	$U(2)\oplus D_4^{\oplus 2}$	$\surd$
5	5	3	(2,0,0)	2	3	0	1			$\surd$
8	6	2	(1,1,2)	0	2	0	1	3	$U\oplus E_8\oplus D_4$	X
12	2	2	(3,3,4)	(4-0)	(0-2)	1	0			X
10	4	2	(5,1,0)	4	0	1	1			X
14	0	2	(7,3,2)	4	0	2	0			$\surd$
8	6	2	(1,1,2)	2	2	0	1	2	$U(2)\oplus E_8\oplus D_4$	$\surd$
12	2	2	(3,3,4)	2	2	1	0			$\surd\surd$

Table C.4: The case $m=0, g_{\sigma^4}>1$.

m_1	r	N'	N	(n_2,n_3,n_4)	g_{σ^4}	k	$S(\sigma^4)$	Example
4	6	4	6	(5,1,0)	7	1	$U\oplus D_4$	$\surd$
4	6	6	6	(5,1,0)	6	1	$U(2)\oplus D_4$	$\surd$
2	14	4	12	(7,3,2)	3	2	$U\oplus D_4\oplus E_8$	$\surd$

Table C.5: The case rk Pic(X) $=$ S(σ) $=$ r .

m_1	m	r	l	(n_2, n_3, n_4)	N	N'	k	a	s	g_{σ^2}	$S(\sigma^4)$	Example
3	3	3	1	$(1, 1, 2)$	4	2	0	0	0	2	$U \oplus A_1^{\oplus 8}$	X
2	4	4	2	$(1, 1, 2)$	4	2	0	0	1	2	$U \oplus A_1^{\oplus 4} \oplus E_8$	X
											$U \oplus D_4 \oplus D_8$	X

Table C.6: The case $g_{\sigma^2} > 1$.

m	r	l	N	N'	(n_2, n_3, n_4)	a	k	A	h	s	k_{σ^2}	N'_{σ^2}	g_{σ^4}	Example
1	13	3	10	2	(3,3,4)	0	1	0	2	1	3	6	2	X
1	7	1	6	4	(5,1,0)	0	1	0	0	0	1	4	3	X
2	8	2	6	4	(5,1,0)	0	1	0	0	0	1	4	3	X

Table C.7: The case $g_{\sigma^4} > 1$, $C \not\subseteq \mathrm{Fix}(\sigma^2)$.

m	r	l	N	(n_2, n_3, n_4)	k	k_{σ^2}	N_{σ^2}	Example
1	13	3	10	(3,3,4)	1	3	10	$\surd$

Table C.8: The case $\mathrm{Fix}(\sigma^4)$ has only rational curves, $k > 0$.

m	r	l	s	a	A	h	N	N'	(n_2, n_3, n_4)	k_{σ^2}	N'_{σ^2}	g_{σ^4}	Example
1	2	2	0	0	0	1	2	2	(2,0,0)	0	4	5	X
1	9	7	1	1	1	2	4	2	(1,1,2)	3	6	2	X
3	7	5	2	0	0	2	4	2	(1,1,2)	1	6	2	X
1	6	6	1	1	0	3	2	2	(2,0,0)	2	8	3	X
2	6	4	1	0	0	2	4	0	(1,1,2)	1	4	3	X
1	8	4	1	0	0	3	6	2	(0,2,4)	2	8	3	X
1	5	3	0	0	1	0	4	0	(1,1,2)	1	0	5	X
			0	0	1	0	4	2	(1,1,2)	1	2	4	X
			0	0	0	2	4	0	(1,1,2)	1	4	3	X
			0	0	0	2	4	2	(1,1,2)	1	6	2	X
2	6	4	0	0	1	0	4	0	(1,1,2)	1	-	0	X
3	7	5	1	0	1	0	4	0	(1,1,2)	1	-	0	X
1	9	7	0	1	2	0	4	0	(1,1,2)	3	-	0	$\surd$

Table C.9: The case $k = \alpha = 0, m > 0$.

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