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École doctorale : Sciences et ingénierie pour l'information, mathématiques - S2IM (Poitiers)
Secteur de recherche : Mathématiques et leurs interactions

Présentée par :
Haydi Israel

Comportement asymptotique de modèles en séparation de phases

Directeur(s) de Thèse :
Alain Miranville, Madalina Petcu

Soutenue le 05 décembre 2013 devant le jury

Jury :

Président	**Mikhael Balabane**	Professeur des Universités, Université de Paris 13
Rapporteur	**Pavel Krejči**	Professor, Prague Institute of Mathematics AS CR, Czech Republic
Rapporteur	**Maurizio Grasselli**	Professore Politecnique di Milano, Italia
Membre	**Alain Miranville**	Professeur des Universités, Université de Poitiers
Membre	**Madalina Petcu**	Maître de conférences, Université de Poitiers
Membre	**Jean-Michel Rakotoson**	Professeur des Universités, Université de Poitiers
Membre	**Frédéric Pascal**	Professeur des Universités, ENS de Cachan
Membre	**Laurence Cherfils**	Maître de conférences, Université de La Rochelle

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Directeur de thèse : **Alain Miranville**
Co-directeur de thèse : **Madalina Petcu**

Soutenue le Jeudi 05 Décembre 2013
devant la commission d'Examen

Jury

Maurizo Grasselli	Professeur, Politecnique de Milan	Rapporteur
Pavel Krejci	Professeur, Institut AS CR	Rapporteur
Mikhael Balabane	Professeur, Université Paris 13	Examinateur
Laurence Cherfils	MCF Habilité, Université de La Rochelle	Examinateur
Frédéric Pascal	Professeur, ENS de Cachan	Examinateur
Jean-Michel Rakotoson	Professeur, Université de Poitiers	Examinateur
Alain Miranville	Professeur, Université de Poitiers	Directeur de thèse
Madalina Petcu	MCF Habilité, Université de Poitiers	co-directeur de thèse

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Table des matières

Remerciements v
Table des matières vii

Introduction **1**

1 Well-Posedness and Long Time Behavior of an Allen-Cahn Type Equation **7**
1.1 Introduction 9
1.2 Existence and uniqueness results 11
1.2.1 A priori estimates 12
1.2.2 Passage to the limit 14
1.2.3 Uniqueness 15
1.3 Additional regularity 16
1.4 Existence of the global attractor 19
1.5 Existence of an exponential attractor 24

2 Well-Posedness and Long Time Behavior of a Perturbed Cahn-Hilliard System with Regular Potentials **29**
2.1 Introduction 31
2.2 Notation and assumptions 32
2.3 Uniform a priori estimates. Existence and uniqueness of solutions. . . 35
2.4 Additional regularity 41
2.5 Existence of the global attractor 47
2.6 Existence of an exponential attractor 51
2.7 Construction of a robust family of exponential attractors 55

3 A Cahn-Hilliard Type Equation With Dynamic Boundary Conditions and Regular Potentials **63**
3.1 Introduction 63
3.1.1 Assumptions and notations 64
3.2 Uniform a priori estimates 65
3.3 Existence of solutions. 73
3.4 Global and exponential attractor 75

4 Numerical Analysis of a Cahn-Hilliard Type Equation With Dynamic Boundary Conditions **81**
4.1 Introduction 81
4.2 Assumptions and notation 82

4.3 The semi-discrete scheme 83
4.4 Error estimates for the space semi-discrete scheme 87
4.5 Stability of the backward Euler scheme 96
4.6 Numerical simulations 100

5 Long Time Behavior of an Allen-Cahn Type Equation With a Singular Potential and Dynamic Boundary Conditions 103
5.1 Introduction 105
5.2 Approximations and uniform a priori estimates 106
5.3 Variational formulation and well-posedness 116
5.4 Additional regularity results and separation from the singularities . . 123
5.5 Attractors and exponential attractors 127

Bibliographie 135

Introduction

Cette thèse réunit un certain nombre de résultats théoriques et numériques relatifs à un problème de type Cahn-Hilliard. Ces résultats portent sur le caractère bien posé du problème ainsi que sur l'étude du comportement asymptotique des solutions en termes d'existence de l'attracteur global et d'attracteurs exponentiels.

• Présentation du problème :

Dans cette thèse, on considère le problème de type Cahn-Hilliard suivant :

$$\begin{cases} \partial_t u + \Delta^2 u - \Delta u - \Delta f(u) + f(u) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

où Ω est un domaine régulier et borné, f est une fonction non linéaire et u est le paramètre d'ordre qui représente la concentration locale de l'un des deux composants du mélange.
En posant $w = -\Delta u + f(u)$, le problème (1) peut être formulé comme étant un système de deux équations de second ordre :

$$\begin{cases} \partial_t u = \Delta w - w, \\ w = -\Delta u + f(u), \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

Ce problème a été introduit récemment et étudié dans [30], [31], [32], [37]. Il représente une simplification d'un modèle mésoscopique des mécanismes microscopiques dans les processus de surface tels que la diffusion de surface et l'adsorption-désorption. Le modèle mésoscopique correspondant est une combinaison de la dynamique d'Arrhenius adsorption/désorption, de la diffusion Metropolis de surface et d'une simple réaction unimoléculaire. Il est décrit par :

$$\partial_t u - D\nabla\cdot[\nabla u - \beta u(1-u)\nabla J_m * u] - [k_a p(1-u) - k_d u\exp(-\beta J_d * u)] + k_r u = 0, \tag{3}$$

où $D > 0$ représente la constante de diffusion, k_r, k_d et k_a désignent respectivement les constantes de réaction, de desorption et d'adsorption, p est la pression partielle de l'espèce gazeuse, J_d et J_m sont les potentiels inter-moléculaires de la désorption et la migration de surface. Pour plus de détails, on réfère [30].

On note que l'on peut voir comme une combinaison de l'équation de Cahn-Hilliard

$$\partial_t u = -\Delta^2 u + \Delta f(u), \quad u(0,x) = u_0(x)$$

et de l'équation d'Allen-Cahn

$$\partial_t u = \Delta u - f(u), \quad u(0,x) = u_0(x).$$

L'équation de Cahn-Hilliard a été très étudiée, tant d'un point de vue mathématique que d'un point de vue numérique (voir [7], [10], [14], [42]). Ce modèle décrit le processus de séparation de phase dans un alliage binaire, en particulier, la décomposition spinodale. Un tel processus peut être observé lorsqu'un alliage binaire, de composition homogène à température élevée, avec une concentration uniforme de chacune des deux phases, est brutalement refroidi; le matériau devient non homogène, les phases se séparant en domaines de concentration relativement plus élevée en l'une ou l'autre des phases.

De même, l'équation d'Allen-Cahn, aussi connue des mathématiciens sous le nom d'équation de la chaleur semi-linéaire, a été étudiée du point de vue mathématique et numérique (voir [1], [53], [57]). Cette équation est fondamentale en science des matériaux et décrit un modèle simplifié d'adsorption et de désorption à partir d'une surface.

Dans cette thèse, deux différents types de conditions au bord ont été proposés et pour chaque type on donne deux conditions sur le bord vu que (1) est une équation du quatrième ordre en espace. On propose également différents types de nonlinéarités.

D'abord, on considère des conditions de type Dirichlet, typiquement :

$$u = \Delta u = 0, \text{ sur le bord } \partial\Omega.$$

On désigne par $\mathcal{E}$ l'énergie de l'équation, elle est donnée par :

$$\mathcal{E}(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u))\mathrm{d}x,$$

où le potentiel F est une primitive de f avec $f(0) = 0$. On remarque que $\mathcal{E}$ est une fonctionnelle de Lyapunov c'est à dire que $\mathcal{E}(u(t))$ décroît avec le temps t. En effet,

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\int_\Omega (|\nabla w|^2 + |w|^2)\mathrm{d}x, \ t \geqslant 0.$$

Ensuite, on considère des conditions dynamiques sur le bord, typiquement,

$$\partial_n w = 0, \ x \in \partial\Omega, \tag{4}$$

$$\partial_t u = \Delta_\Gamma u - g(u) - \lambda u - \partial_n u, \ x \in \partial\Omega, \tag{5}$$

où Δ_Γ est l'opérateur de Laplace-Beltrami sur le bord et ∂_n est la dérivée normale extérieure. Cette appelation vient du fait que la dérivée de u par rapport au temps, $\partial_t u$, apparaît explicitement dans (5).

La condition (5) a été récemment introduite par les physiciens afin de tenir compte des interactions entre les deux constituants du mélange et la paroi (le bord $\partial\Omega$) dans les systèmes confinés et décrire l'influence de cette dernière sur le processus de séparation de phases. Cependant, la condition (4) signifie qu'il ne peut y avoir aucun échange des constituants du mélange à travers la paroi. Pour plus de détails

sur l'équation de Cahn-Hilliard classique avec des conditions dynamiques sur le bord, on réfère [19], [33], [48], [49]. Avec ce type de conditions, l'énergie libre de l'équation est donnée par :

$$\mathcal{E}(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) \mathrm{d}x + \int_{\partial\Omega} \left(\frac{1}{2}|\nabla_\Gamma u|^2 + \frac{\lambda}{2}|u|^2 + G(u)\right) \mathrm{d}\sigma,$$

où G est une primitive de g. La première intégrale est l'énergie de volume du matériau et la seconde intégrale est l'énergie de surface. De même, on a une dissipation d'énergie,

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\int_\Omega (|\nabla w|^2 + |w|^2)\mathrm{d}x - \int_{\partial\Omega} |\partial_t u|^2 \mathrm{d}\sigma,\ t \geqslant 0.$$

Outre l'existence et l'unicité des solutions, on établit le comportement asymptotique des solutions. On cherche l'attracteur global qui, lorsqu'il existe, est l'unique ensemble compact, invariant par le semi-groupe, qui attire toutes les solutions du problème lorsque le temps t tend vers $+\infty$ et qui est minimal parmi les ensembles vérifiant cette définition.
Cependant, l'attracteur global est sensible aux perturbations et cela est lié à la vitesse d'attraction de trajectoires qui peut être lente. Dans le but de corriger ce défaut, on a introduit dans [9] la notion d'attracteur exponentiel.
Par définition, un attracteur exponentiel est un ensemble compact qui contient l'attracteur global, qui est beaucoup plus robuste face aux perturbations, qui est de dimension fractale finie et qui attire à vitesse exponentielle toutes les solutions du problème. Il a été d'abord construit en démontrant la propriété de laminage qui n'est valable que dans des espaces de Hilbert et n'est pas vraie dans des espaces de Banach, voir [1], [9]. Récement, on donne dans [10] une construction d'attracteurs exponentiels plus générale, pour des espaces de Banach, construction qui consiste à vérifier une propriété de régularisation sur la différence de deux solutions.
Vu qu'un attracteur exponentiel est robuste face aux perturbations, on propose dans cette thèse d'étudier également son existence. Une fois l'existence établie, on en déduit que la dimension fractale de l'attracteur global est finie.

• Présentation des résultats et plan de la thèse :

La thèse est organisée comme suit :

Dans le premier chapitre, on considère le problème (1) avec des conditions aux limites de type Dirichlet et une nonlinéarité f régulier, typiquement,

$$f(s) = \sum_{i=1}^{2p-1} a_i s^i,\ p \in \mathbb{N},\ p \geqslant 2,\ a_{2p-1} > 0.$$

On démontre l'existence et l'unicité de la solution faible qui correspond à une donnée initiale dans $H_0^1(\Omega) \cap L^{2p}(\Omega)$ et l'existence de la solution forte quand la donnée initiale est dans $H^2(\Omega) \cap H_0^1(\Omega)$. De plus, on étudie le comportement asymptotique des solutions et on démontre l'existence de l'attracteur global dans $H^{-1}(\Omega)$ qui est borné

dans $H^2(\Omega) \cap H_0^1(\Omega)$. On prouve également l'existence d'un attracteur exponentiel en vérifiant une propriété de régularisation $H^{-1}(\Omega) - L^2(\Omega)$ sur la difference de deux solutions.

Dans le deuxième chapitre, on considère une équation perturbée du problème (1) avec le même type de conditions aux limites du deuxième chapitre :

$$\begin{cases} \partial_t u + \Delta^2 u - \Delta f(u) - \varepsilon \Delta u + \varepsilon f(u) = 0, & \text{dans } \Omega, \\ u = \Delta u = 0 & \text{sur } \partial\Omega, \\ u|_{t=0} = u_0, \end{cases} \tag{6}$$

où $\varepsilon \geqslant 0$. La fonction f satisfait cette fois ci les conditions suivantes :

$$\begin{cases} f \in C^2(\mathbb{R}), \\ f(0) = 0, \\ f'(s) \geqslant -\kappa, \\ f(s)s \geqslant p b_{2p} s^{2p} - c \geqslant c(s^2 + s^{2p}) - c, \\ |f(s)| \leqslant c(|s|^{2p-1} + 1). \end{cases} \tag{7}$$

En ce qui concerne le caractère bien posé du problème (6), la régularité des solutions et l'existence de l'attracteur global, on a des résultats analogues à ceux du premier chapitre. Cependant, on prouve l'existence d'un attracteur exponentiel en vérifiant une propriété de régularisation $H^{-1}(\Omega) - H^2(\Omega)$ sur la difference de deux solutions. À la fin de ce chapitre, on démontre l'existence d'une famille robuste d'attracteurs exponentiels, on étudie la limite lorsque ε tend vers 0 et on prouve la continuité des attracteurs exponentiels du problème perturbé vers un attracteur exponentiel du problème non perturbé qui n'est autre que le problème de Cahn-Hilliard classique.

Dans les chapitres 3 et 4, on donne une étude théorique et numérique pour le problème (1) avec des conditions aux limites dynamiques et des nonliéarités régulières :

$$\begin{cases} \partial_t u = \Delta w - w, \ \partial_n w = 0, \ x \in \partial\Omega \\ w = -\Delta u + f(u), \ u_{|t=0} = u_0, \\ \partial_t v = \Delta_\Gamma v - g(v) - \lambda v - \partial_n u, \ x \in \partial\Omega, \ v_{|t=0} = v_0 \\ u_{|\partial\Omega} = v. \end{cases} \tag{8}$$

Les fonctions f et g sont dans $C^2(\mathbb{R})$ satisfaisant les conditions de dissipativité suivantes :

$$\liminf_{|s|\to+\infty} f'(s) > 0, \ \liminf_{|s|\to+\infty} g'(s) > 0. \tag{9}$$

D'abord, on démontre l'existence et l'unicité de la solution $(u(t), v(t))$ dans $L^\infty([0,T], \mathbb{W})$ avec $(\partial_t u, \partial_t v)$ dans $L^2([0,T], \mathbb{V})$ où

$$\mathbb{W} := \{(u,v) \in H^2(\Omega) \times H^2(\Gamma), \ w = -\Delta u + f(u) \in H^1(\Omega), \ u_{|\Gamma} = v, \ \partial_n w_{|\Gamma} = 0\}$$

et

$$\mathbb{V} := \{(u,v) \in H^1(\Omega) \times H^1(\Gamma), \ u_{|\Gamma} = v\}.$$

De plus, on démontre l'existence de l'attracteur global $\mathcal{A} \subset \mathbb{W} \cap H^3(\Omega) \times H^3(\Gamma)$ et également l'existence d'un attracteur exponentiel en vérifiant une propriété de régularisation $H^{-1}(\Omega) \times L^2(\Gamma) - H^2(\Omega) \times H^2(\Gamma)$.

Ensuite, on effectue l'analyse numérique du problème. On considère un domaine Ω correspondant à des conditions aux limites périodiques telles que :

$$\Omega = \Pi_{i=1}^{d-1}(\mathbb{R}/(L_i\mathbb{Z})) \times (0, L_d),\ L_i > 0, i = 1, ..., d,\ d = 2 \text{ or } 3,$$

avec la frontière régulière

$$\Gamma = \partial\Omega = \Pi_{i=1}^{d-1}(\mathbb{R}/(L_i\mathbb{Z})) \times \{0, L_d\}\,.$$

On propose une discrétisation en espace par des éléments finis et on démontre l'existence et l'unicité de la solution de la version discrète associée à la formulation variationnelle de (8). Puis, on prouve des estimations d'erreur optimales dans certaines normes pour la différence $u^h - u$ entre la solution approchée de la version discrète et la solution exacte du problème continu lorsque le pas de maillage h tend vers 0.
On aborde aussi l'étude du problème totalement discrétisé. Pour la discrétisation en temps et en espace, on utilise un schéma d'Euler implicite en temps et semi-discrétisé en espace et on démontre l'existence, l'unicité et la stabilité de la solution. Ces résultats sont illustrés par des simulations numériques en dimension deux d'espace réalisées avec FreeFem++, simulations qui permettent d'étudier l'influence des différents paramètres.

Dans le dernier chapitre, on considère (1) avec des conditions dynamiques sur le bord et f singulier :

$$\begin{cases} \partial_t u = \Delta w - w,\ \partial_n w = 0,\ x \in \partial\Omega, \\ w = -\Delta u + f(u) - \lambda u,\ u_{|t=0} = u_0, \\ \partial_t v = \Delta_\Gamma v - g(v) - \partial_n u,\ x \in \partial\Omega,\ v_{|t=0} = v_0, \\ u_{|\Gamma} = v. \end{cases} \tag{10}$$

Les fonctions f et g satisfont respectivement

$$\begin{cases} f \in C^2((-1,1)), \\ f(0) = 0,\ \lim\limits_{s\to\pm1} f(s) = \pm\infty, \\ f'(s) \geqslant 0,\ \lim\limits_{s\to\pm1} f'(s) = +\infty, \\ f''(s)\ \text{sgn}\ s \geqslant 0, \end{cases} \tag{11}$$

$$g(\sigma) = \sigma + g_0(\sigma), \quad \forall\sigma \in \mathbb{R},\ \text{où}\ \|g_0\|_{C^2(\mathbb{R})} := C_0 < +\infty. \tag{12}$$

Tout d'abord, on approche f par des fonctions régulières et on obtient des estimations à priori uniformes sur les solutions correspondantes. Ensuite, on définit la solution variationnelle pour le problème et on étudie l'existence de la solution au sens usuel. On termine ce chapitre par le comportement asymptotique des solutions.

Chapitre 1

Well-Posedness and Long Time Behavior of an Allen-Cahn Type Equation

Sur le caractère bien posé et le comportement asymptotique d'une équation de type Allen-Cahn

Ce chapitre est constitué de l'article **Well-posedness and long time behavior of an Allen-Cahn type equation**, paru en 2013 dans *Communication on Pure and Applied Analysis*, volume 12, numero 6, pages 2811-2827.

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Well-posedness and long time behavior of an Allen-Cahn type equation

Haydi ISRAEL
UMR 7348 CNRS. Laboratoire de Mathématiques et Applications - Université de Poitiers
SP2MI - Boulevard Marie et Pierre Curie - Téléport 2
BP30179 - 86962 Futuroscope Chasseneuil Cedex - FRANCE.

Abstract. The aim of this article is to study the existence and uniqueness of solutions for an equation of Allen-Cahn type and to prove the existence of the finite-dimensional global attractor as well as the existence of exponential attractors.

1.1 Introduction

In this article we are interested in the study of the following partial differential equation, considered in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$:

$$\begin{cases} \partial_t u + \Delta^2 u - \Delta f(u) - \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a polynomial of order $2p-1$

$$f(s) = \sum_{i=1}^{2p-1} a_i s^i, \quad p \in \mathbb{N}, \ \ p \geqslant 2.$$

This equation is associated with the effect of multiple microscopic mechanisms such as surface diffusion and adsorption/desorption and it was recently derived and studied in [30], [32], [37]. We note that equation (1.1) may be viewed as a combination of the well-known Cahn-Hilliard equation

$$\partial_t u = -\Delta(\Delta u - f(u)), \quad u(0,x) = u_0(x)$$

and of the Allen-Cahn equation

$$\partial_t u = \Delta u - f(u), \quad u(0,x) = u_0(x).$$

We recall that the Cahn-Hilliard equation describes the behavior of two-phase systems, in particular in spinodal decomposition, meaning in the case of rapid separations of phases when the material is cooled down sufficiently. The Allen-Cahn

equation is also used in the study of two-phase systems and it describes the ordering of atoms within unit cells on a lattice.
We denote by F the primitive of f vanishing at $u = 0$,

$$F(s) = \sum_{j=2}^{2p} b_i s^j, \quad jb_j = a_{j-1}, \quad 2 \leqslant j \leqslant 2p,$$

and we assume that the leading coefficient of f (and g) is positive

$$a_{2p-1} = 2pb_{2p} > 0.$$

Since $a_{2p-1} > 0$, it is easy to conclude that there exists two constants c_1 and c_2 such that

$$\begin{array}{l} \frac{a_{2p-1}}{2} s^{2p} - c_1 \leqslant f(s)s \leqslant \frac{3}{2} a_{2p-1} s^{2p} + c_1, \ \forall s \in \mathbb{R}; \\ f'(s) \geqslant \frac{a_{2p-1}}{2p} s^{2p-2} - c_2 \geqslant -\kappa, \ \forall s \in \mathbb{R}. \end{array} \tag{1.2}$$

Furthermore, there exists a constant c_3 such that:

$$\frac{1}{4p} a_{2p-1} s^{2p} - c_3 \leqslant F(s) \leqslant \frac{3}{4p} a_{2p-1} s^{2p} + c_3, \quad \forall s \in \mathbb{R}. \tag{1.3}$$

For the mathematical setting of the problem, we introduce the following space $H = L^2(\Omega)$, which we endow with the scalar product $(.,.)$ and the norm $|.|$.
Let

$$A_2 = -\Delta : D(A_2) \subset H \to H \text{ and } A_1 = -\Delta + I : D(A_1) \subset H \to H$$

with $D(A_1) = D(A_2) = W := \left\{u \in H^2, \ u = 0 \text{ on } \partial\Omega\right\}$. A_1 is a strictly positive self-adjoint linear operator with compact inverse A_1^{-1}. We set $V = D(A_2^{1/2})$ and we endow V with the scalar product $((.,.)) = (A_2^{1/2}., A_2^{1/2}.)$ and the norm $\|.\| = |A_2^{1/2}.|$. More generally, we endow $D(A_2^s), \ s \in \mathbb{R}$, with the scalar product $((.,.))_s = (A_2^s., A_2^s.)$ and the norm $\|.\|_s = |A_2^s.|$.
Then, problem (1.1) can be reformulated as follows:

$$\left\{ \begin{array}{ll} A_1^{-1}\partial_t u + A_2 u + f(u) = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.4}$$

The variational formulation of problem (1.4) reads:
Find $u : [0, T] \to V$ *such that*

$$\left\{ \begin{array}{l} \left(A_1^{-1}\partial_t u, q\right) + (A_2 u, q) + (f(u), q) = 0, \ \forall q \in V, \\ u(0) = u_0, \end{array} \right. \tag{1.5}$$

$\forall T > 0$.
Our aim is to prove the existence and uniqueness of solutions, as well as the existence of a finite-dimensional attractor for the proposed model (1.1). This article is structured as follows. In section 2 we consider problem (1.1) with Dirichlet boundary conditions and we prove the existence and uniqueness of solutions. Section 3 is dedicated to the proof of the existence of the global attractor and in Section 4 we prove a stronger result, namely the existence of an exponential attractor, which implies the finite-dimensionality of the global attractor.

1.2 Existence and uniqueness results

In this section, we establish the existence and uniqueness of a solution for problem (1.4). The proof is based on a priori estimates and on the Faedo-Galerkin scheme. To do this, let us first consider the eigenfunctions $\{e_j\}_j$ of the operator A_1, $A_1 e_j = \lambda_j e_j$ with $e_j \in V$ for all $j \in \mathbb{N}$. We know that the eigenfunctions e_j form an orthogonal basis in H and V and the family of $\{e_j\}_j$ may be assumed to be normalized in the norm of H, i.e.,

$$(e_i, e_j) = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by E_m the space

$$E_m = \text{span}\{e_1, e_2, \ldots, e_m\},$$

and by P_m the orthogonal projection from V onto E_m:

$$P_m h = \sum_{j=1}^{m} (h, e_j)\, e_j.$$

For any $m \in \mathbb{N}$, we look for functions of the form

$$u_m(t) = \sum_{j=1}^{m} u_{m_j}(t)\, e_j,$$

solving the approximate problem below:

$$\begin{cases} \dfrac{d}{dt}\left(A_1^{-1}\displaystyle\sum_{j=1}^{m} u_{m_j}(t)\, e_j, e_i\right) + \left(A_2 \displaystyle\sum_{j=1}^{m} u_{m_j}(t)\, e_j, e_i\right) + (f(u_m), e_i) = 0,\ i = 1, ..., m, \\ u_m(0) = P_m u_0. \end{cases} \tag{1.6}$$

We rewrite problem (1.6) as follows:

$$M_1 \frac{dY}{dt} + M_2 Y + F(Y) = 0,$$

where $M_1 = \left(\left(A_1^{-1}(e_i), e_j\right) \right)_{i,j=1,\ldots,m}$, $M_2 = \left((A_2(e_i), e_j) \right)_{i,j=1,\ldots,m}$,

$Y = \begin{pmatrix} u_{m_1} \\ \vdots \\ u_{m_m} \end{pmatrix}$ and $F(Y) = \begin{pmatrix} (f(u_m), e_1) \\ \vdots \\ (f(u_m), e_m) \end{pmatrix}$.

We can easily check that matrix M_1 is invertible. Indeed, setting $X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ and $X' = \displaystyle\sum_{i=1}^{m} x_i e_i$, we have

$$(M_1 X, X) = \sum_{j=1}^{m}\sum_{i=1}^{m} \left(A_1^{-1} e_j, e_i\right) x_i x_j = \left(A_1^{-1} X', X'\right) > 0,$$

due to the fact that A_1^{-1} is a positive definite operator. Moreover, the matrix M_2 is positive definite and $F(Y)$ depends continuously on Y. Applying Cauchy's theorem for a system of ordinary differential equations, we find that there exists a time $t_m \in (0,T)$ and a unique solution Y for the equation $\frac{dY}{dt}+M_1^{-1}M_2Y+M_1^{-1}F(Y)=0$ on the time interval $t \in [0,t_m[$. Based on the a priori estimates with respect to t that will be derived below for the solution $u_m(t)$, we obtain that any local solution of (1.6) is actually a global solution that is defined on the whole interval $[0,T]$.
Now, we give the a priori estimates for the solution $u_m(t)$:

1.2.1 A priori estimates

The solution $u_m(t)$ verifies the following approximate problem:

$$\begin{cases} A_1^{-1}\partial_t u_m + A_2 u_m + f(u_m) = 0, \\ u_m(0) = P_m u_0. \end{cases} \tag{1.7}$$

Multiplying equation (1.7) by $A_1 u_m(t)$ and integrating over Ω, we obtain:

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + |A_2 u_m(t)|^2 + |A_2^{1/2}u_m(t)|^2 + \int_\Omega f(u_m)\cdot A_1 u_m(t)\mathrm{d}x = 0.$$

Using the fact that

$$\int_\Omega f(u_m(t))\cdot u_m(t)\mathrm{d}x \geqslant \frac{a_{2p-1}}{2}\int_\Omega u_m^{2p}(t)\mathrm{d}x - c_1|\Omega|,$$

and

$$\begin{aligned}(f(u_m(t)), A_2u_m(t)) =& (f'(u_m(t))A_2^{1/2}u_m(t), A_2^{1/2}u_m(t)) \\ \geqslant& \frac{a_{2p-1}}{2p}\int_\Omega u_m^{2p}(t)|A_2^{1/2}u_m(t)|^2\mathrm{d}x - c_2\int_\Omega |A_2^{1/2}u_m(t)|^2\mathrm{d}x,\end{aligned}$$

we find:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + &|A_2u_m(t)|^2 + |A_2^{1/2}u_m(t)|^2 + \frac{a_{2p-1}}{2}\int_\Omega u_m^{2p}(t)\mathrm{d}x \\ &+ \frac{a_{2p-1}}{2p}\int_\Omega u_m^{2p}(t)|A_2^{1/2}u_m(t)|^2\mathrm{d}x \\ \leqslant& c_1|\Omega| + c_2\int_\Omega |A_2^{1/2}u_m(t)|^2\mathrm{d}x \leqslant c_1|\Omega| + c_2\left(A_2u_m(t), u_m(t)\right) \\ \leqslant& c_1|\Omega| + \frac{1}{2}|A_2u_m(t)|^2 + c_3'|u_m(t)|^2.\end{aligned} \tag{1.8}$$

From (1.8), we deduce:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + &\frac{1}{2}|A_2u_m(t)|^2 + |A_2^{1/2}u_m|^2 + \frac{a_{2p-1}}{2}\int_\Omega u_m^{2p}(t)\mathrm{d}x \\ &+ \frac{a_{2p-1}}{2p}\int_\Omega u_m^{2p}(t)|A_2^{1/2}u_m(t)|^2\mathrm{d}x \\ \leqslant& c_1|\Omega| + c_3'|u_m(t)|^2.\end{aligned} \tag{1.9}$$

In particular, we have

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 \leqslant c_1|\Omega| + c_3^{'}|u_m(t)|^2. \tag{1.10}$$

Consequently, applying Gronwall's inequality to (1.10), we find:

$$\begin{aligned} |u_m(t)|^2 &\leqslant \left(|u_0|^2 + \frac{c_1}{c_3^{'}}|\Omega|\right) e^{c_3^{'}T} \\ &\leqslant c(u_0, T), \end{aligned}$$

where $c(u_0, T)$ denotes a constant depending on the time T and the initial datum u_0 but independent of m. Then, $\sup\limits_t |u_m(t)|^2$ is bounded independently of m, which implies the global existence of the solutions.
Integrating equation (1.9) with respect to t, from 0 to t, we obtain:

$$\begin{aligned} \frac{1}{2}|u_m(t)|^2 + \frac{1}{2}\int_0^t |A_2 u_m(s)|^2 \mathrm{d}s + \int_0^t |A_2^{\frac{1}{2}} u_m(s)|^2 \mathrm{d}s + \frac{a_{2p-1}}{2}\int_0^t \int_\Omega u_m^{2p}(s)\mathrm{d}x\mathrm{d}s \\ + \frac{a_{2p-1}}{2p}\int_0^t \int_\Omega u_m^{2p}(s)|A_2^{1/2}u_m(s)|^2 \mathrm{d}x\mathrm{d}s \\ \leqslant c_1|\Omega|T + c_3^{'}\int_0^t |u_m(s)|^2\mathrm{d}s + \frac{1}{2}|u_m(0)|^2 \\ \leqslant c_1|\Omega|T + c_3^{'}T\sup|u_m(t)|^2 + \frac{1}{2}|u(0)|^2. \end{aligned} \tag{1.11}$$

Due to equation (1.11), we conclude that $\{u_m\}_m$ is bounded in $L^2(0,T;W)$, $L^\infty(0,T;H)$ and $L^{2p}(0,T;L^{2p}(\Omega))$, independently of m.
Multiplying (1.7) by $\partial_t u_m$ and integrating over Ω, we have:

$$|A_1^{-1/2}\partial_t u_m(t)|^2 + \frac{1}{2}\partial_t|A_2^{1/2}u_m(t)|^2 + \partial_t\int_\Omega F(u_m(t))\mathrm{d}x = 0. \tag{1.12}$$

Integrating (1.12) with respect of t, we find:

$$\begin{aligned} &\int_0^t |A_1^{-1/2}\partial_t u_m(s)|^2\mathrm{d}s + \frac{1}{2}|A_2^{1/2}u_m(t)|^2 + \int_\Omega F(u_m(t))\mathrm{d}x \\ =&\frac{1}{2}|A_2^{1/2}u_m(0)|^2 + \int_\Omega F(u_m(0))\mathrm{d}x. \end{aligned} \tag{1.13}$$

Using (1.3),(1.13) implies:

$$\begin{aligned} &\int_0^t |A_1^{-1/2}\partial_t u_m(s)|^2\mathrm{d}s + \frac{1}{2}|A_2^{1/2}u_m(t)|^2 + \frac{a_{2p-1}}{4p}\int_\Omega u_m^{2p}(t)\mathrm{d}x \\ \leqslant& 2c_3|\Omega| + \frac{1}{2}|A_2^{1/2}u_m(0)|^2 + \frac{3a_{2p-1}}{4p}\int_\Omega u_m^{2p}(0)\mathrm{d}x. \end{aligned}$$

Consequently, if $u_0 \in V \cap L^{2p}(\Omega)$, we deduce that:

$$u_m(t) \in L^\infty(0,T;V\cap L^{2p}(\Omega))$$

and

$$\partial_t u_m(t) \in L^2(0,T;H^{-1}(\Omega)).$$

1.2.2 Passage to the limit

In this section, we intend to pass to the limit as $m \to \infty$ and study the convergence of the sequence $(u_m)_m$. Let q satisfy $\frac{1}{2p}+\frac{1}{q}=1$, which implies $q<2$. According to the a priori estimates derived in the previous section, we have $\|u_m\|_{L^2(0,T;W)}$ uniformly bounded and consequently, $(u_m)_m$ is bounded in $L^q(0,T;L^q(\Omega))$. Moreover, we have $\|f(u_m)\|_{L^q(0,T;L^q(\Omega))}$ uniformly bounded which implies that $(A_1^{-1}\partial_t u_m)$ is bounded in $L^q(0,T;L^q(\Omega))$. It follows that:

$$\|\partial_t u_m\|_{L^q(0,T;W^{-2,q}(\Omega))} \leqslant C.$$

Starting from this point, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not relabelled. Thus, we observe that weak and weak star compactness results applied to the sequence $\{u_m\}_m$ entail that there exists a function u such that the following properties hold:

$$u_m \to u \text{ weakly in } L^q(0,T;W), \tag{1.14}$$

$$\partial_t u_m \to \partial_t u \text{ weakly in } L^q(0,T;W^{-2,q}(\Omega)), \tag{1.15}$$

as $m \to \infty$. It follows from (1.14), (1.15) and Aubin-Lions compactness theorem, that:

$$u_m \to u \text{ strongly in } L^q(0,T;L^q(\Omega)).$$

Consequently $u_m(t,x) \to u(t,x)$ a.e. $(t,x) \in [0,T]\times\Omega$.
Moreover, we have:

$$\left.\begin{array}{l} u_m(t,x) \to u(t,x) \text{ a.e.} \\ f \text{ is a continuous polynomial function} \end{array}\right\} \Longrightarrow \quad f(u_{m_i}(t,x)) \to f(u(t,x)) \text{ a.e.}$$

$$\left.\begin{array}{l} f(u_{m_i}(t,x)) \to f(u(t,x)) \text{ a.e.} \\ ||f(u_{m_i})||_{L^q(\Omega_T)} \leqslant \text{ constant} \end{array}\right\} \Longrightarrow \quad f(u_{m_i}) \to f(u) \text{ weakly in } L^q(\Omega_T).$$

Finally, we deduce that $A_1^{-1}\partial_t u_m \rightharpoonup A_1^{-1}\partial_t u$ weakly in $L^q(\Omega_T)$. Thus, passing to the limit in (1.7), we obtain:

$$A_1^{-1}\partial_t u + A_2 u + f(u) = 0, \quad \text{ in } L^q(\Omega_T). \tag{1.16}$$

We also need to prove that $u(0)=u_0$. To do this, we consider a test function $\psi \in C^1([0,T];L^{2p}(\Omega))$ such that $\psi(T)=0$. Multiplying (1.16) by ψ and integrating over $\Omega\times[0,T]$, we obtain:

$$\int_0^T \left\langle A_1^{-1}\partial_t u, \psi\right\rangle dt + \int_0^T \langle A_2 u, \psi\rangle\, dt + \int_0^T \langle f(u), \psi\rangle\, dt = 0. \tag{1.17}$$

Integrating by parts in (1.17), we have:

$$-\int_0^T \left\langle A_1^{-1}u, \partial_t\psi\right\rangle dt - \left\langle A_1^{-1}u(0), \psi(0)\right\rangle + \int_0^T \langle A_2 u, \psi\rangle\, dt + \int_0^T \langle f(u), \psi\rangle\, dt = 0. \tag{1.18}$$

Multiplying (1.7) by ψ and integrating over $\Omega \times [0,T]$, we obtain:

$$-\int_0^T \left\langle A_1^{-1}u_m, \partial_t\psi \right\rangle dt - \left\langle A_1^{-1}P_m u(0), \psi(0) \right\rangle + \int_0^T \langle A_2 u_m, \psi\rangle \, dt + \int_0^T \langle f(u_m), \psi\rangle \, dt = 0. \tag{1.19}$$

Having $\psi \in L^{2p}(0,T;L^{2p}(\Omega))$ and $\dfrac{\partial\psi}{\partial t} \in L^2(0,T;H)$ we deduce that:

$$\int_0^T \langle A_2 u_m, \psi\rangle \, dt \to \int_0^T \langle A_2 u, \psi\rangle \, dt,$$

and

$$\int_0^T \left\langle A_1^{-1}u_m, \partial_t\psi \right\rangle dt \to \int_0^T \left\langle A_1^{-1}u, \partial_t\psi \right\rangle dt.$$

Then, passing to the limit in (1.19) as $n \to \infty$, we obtain:

$$-\int_0^T \left\langle A_1^{-1}u, \partial_t\psi \right\rangle dt - \left\langle A_1^{-1}u_0, \psi(0) \right\rangle + \int_0^T \langle A_2 u, \psi\rangle \, dt + \int_0^T \langle f(u), \psi\rangle \, dt = 0. \tag{1.20}$$

We deduce from (1.18) and (1.20) that $A_1^{-1}u(0) = A_1^{-1}u_0$, which implies $u(0) = u_0$.

1.2.3 Uniqueness

In what follows we need to prove the uniqueness of solutions for equation (1.4). Let u and v be two solutions of (1.4) on the time interval $[0,T]$. We set $w = u - v$, w verifies the following equation:

$$\begin{cases} \left(A_1^{-1}\partial_t w, q\right) + (A_2 w, q) + (f(u) - f(v), q) = 0, \ \forall q \in W, \\ w(0) = u(0) - v(0). \end{cases} \tag{1.21}$$

Multiplying (1.21) by w and integrating over Ω, we obtain:

$$\frac{1}{2}\frac{d}{dt}|A_1^{-1/2}w|^2 + |A_2^{1/2}w|^2 + (f(u) - f(v), w) = 0. \tag{1.22}$$

Using the fact that $f'(s) \geqslant -\kappa$ and $f(u) - f(v) = l(t)w$, with l defined as:

$$l(t) = \int_0^1 f'\left(su(t) + (1-s)v(t)\right) \mathrm{d}s,$$

we have:

$$(f(u) - f(v), w) \geqslant -\kappa |w|^2.$$

Using the following interpolation inequality

$$|w|^2_{L^2(\Omega)} \leqslant c||w||_{H^{-1}(\Omega)}|A_2^{1/2}w|,$$

we obtain:

$$\frac{1}{2}\frac{d}{dt}||w||^2_{-1} + \frac{1}{2}|A_2^{1/2}w|^2 \leqslant \frac{c^2\kappa^2}{2}||w||^2_{-1}. \tag{1.23}$$

Applying Gronwall's inequality to (1.23), we find:

$$||w(t)||_{-1}^2 \leqslant ||u(0) - v(0)||_{-1}^2 e^{c^2\kappa^2 t}. \tag{1.24}$$

Relation (1.24) shows the continuous dependence of the solution on the initial data and in particular, when $u(0) = v(0)$, it implies the uniqueness of the solution.
Due to previous estimates, we can conclude on the following result:

Theorem 1.2.1. *Let us take $u_0 \in H$. Then, there exists a unique solution u of problem* (1.1) *with initial datum u_0 such that:*

$$u \in L^2([0,T];W) \cap L^\infty([0,T];H).$$

Furthermore, if $u_0 \in V \cap L^{2p}(\Omega)$, then:

$$u \in C([0,T];V \cap L^{2p}(\Omega)) \cap L^2([0,T];W) \text{ and } \partial_t u \in L^2([0,T];H^{-1}(\Omega)).$$

1.3 Additional regularity

In this section, we will derive some additionnal regularity for the solution $u(t)$.

Lemma 1.3.1. *Let $u(t)$ be a solution of* (1.4) *with $u_0 \in W$. Then, there exists a time $T_0 = T_0(\|u_0\|_{H^2(\Omega)})$, $0 < T_0 < 1/2$, and a monotonic function Q such that:*

$$|A_2 u(t)| \leqslant Q(\|u_0\|_{H^2(\Omega)}), \quad t \leqslant T_0(\|u_0\|_{H^2(\Omega)}). \tag{1.25}$$

Proof:
We rewrite equation (1.4) in the following equivalent form:

$$\frac{du}{dt} + (I + A_2)A_2 u + (I + A_2)f(u) = 0,\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0. \tag{1.26}$$

Multiplying equation (1.26) by $A_2^2 u(t)$, we obtain the following inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A_2 u(t)|^2 + |A_2^{3/2}u(t)|^2 + |A_2^2 u|^2 =& -\left((I + A_2)f(u(t)), A_2^2 u(t)\right)\\ \leqslant& \frac{1}{2}|(I + A_2)f(u(t))|^2 + \frac{1}{2}|A_2^2 u(t)|^2\\ \leqslant& \frac{1}{2}\|f(u(t))\|_{H^2(\Omega)}^2 + \frac{1}{2}|A_2^2 u(t)|^2.\end{aligned} \tag{1.27}$$

We recall that $f \in C^2(\Omega)$ and that $H^2(\Omega) \subset C(\overline{\Omega})$. Consequently, there exists a monotonic function Q (depending on f) such that:

$$\|f(u(t))\|_{H^2(\Omega)}^2 \leqslant Q\left(\|u(t)\|_{H^2(\Omega)}^2\right) \leqslant Q_1\left(|A_2 u(t)|^2\right). \tag{1.28}$$

Thus the function $y(t) := |A_2 u(t)|^2$ satisfies the inequality:

$$y'(t) \leqslant Q_1\left(y(t)\right). \tag{1.29}$$

Let $z(t)$ be a solution of the following equation:

$$z'(t) = Q_1\left(z(t)\right),\ z(0) = y(0) = |A_2 u(0)|^2. \tag{1.30}$$

Due to the comparison principle, there exists a time $T_0(\|u_0\|_{H^2(\Omega)}) \in (0, 1/2)$ such that we have:

$$y(t) \leqslant z(t),\ \ \forall t \leqslant T_0(\|u_0\|_{H^2(\Omega)}). \tag{1.31}$$

Then, the lemma is an immediate consequence of (1.30) and (1.31). □

Lemma 1.3.2. *Let the above assumptions hold and let T_0 be the same as in Lemma 1.3.1. Then, the following estimate holds:*

$$t\|\partial_t u(t)\|_{-1}^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}),\ t \in (0, T_0], \tag{1.32}$$

for some monotonic function Q.

Proof:
Multiplying (1.4) by $\partial_t u(t)$ and integrating over Ω, we obtain:

$$\|\partial_t u(t)\|_{-1}^2 + \frac{1}{2}\frac{d}{dt}|A_2^{1/2}u(t)|^2 \leqslant |(f(u(t)), \partial_t u(t))| \leqslant c\|f(u(t))\|_{H^1(\Omega)}^2 + 1/2\|\partial_t u(t)\|_{-1}^2, \tag{1.33}$$

where c is a positive constant. Integrating (1.33) over $[0, T_0]$ and taking into account (1.25) and the fact that $H^2(\Omega) \subset C(\Omega)$, we have:

$$\begin{aligned}\int_0^{T_0} \|\partial_t u(t)\|_{-1}^2 dt \leqslant & c\int_0^{T_0} \|f(u(t))\|_{H^1(\Omega)}^2 dt + |A_2^{1/2}u(0)|^2 \\ \leqslant & c'' T_0 Q(|A_2 u(0)|^2) + |A_2^{1/2}u(0)|^2 \\ \leqslant & Q(\|u_0\|_{H^2(\Omega)}).\end{aligned} \tag{1.34}$$

Differentiating (1.4) with respect to t and setting $\theta(t) = \partial_t u(t)$, we find:

$$\begin{cases} A_1^{-1}\partial_t \theta + A_2\theta + f'(u(t))\theta = 0, \\ \theta|_{\partial\Omega} = 0. \end{cases} \tag{1.35}$$

Multiplying (1.35) by $t\theta(t)$, integrating over Ω and using the fact that $f'(u) \geqslant -\kappa$, we obtain:

$$\begin{aligned}\frac{d}{dt}(t\|\theta(t)\|_{-1}^2) + 2t|A_2^{1/2}\theta(t)|^2 \leqslant & 2\kappa t|\theta(t)|^2 + \|\theta(t)\|_{-1}^2 \\ \leqslant & 2c\kappa t\|\theta(t)\|_{-1}|A_2^{1/2}\theta(t)| + \|\theta(t)\|_{-1}^2 \\ \leqslant & t|A_2^{1/2}\theta(t)|^2 + C(t+1)\|\theta(t)\|_{-1}^2,\end{aligned} \tag{1.36}$$

where C is a positive constant. Hence, we have:

$$\frac{d}{dt}(t\|\theta(t)\|_{-1}^2) \leqslant C(t+1)\|\theta(t)\|_{-1}^2. \tag{1.37}$$

Applying Gronwall inequality to estimate (1.37) and using (1.34), we deduce:

$$t\|\theta(t)\|_{-1}^2 \leqslant \int_0^t \|\theta(s)\|_{-1}^2 \mathrm{d}s \leqslant Q(\|u_0\|_{H^2(\Omega)}), \tag{1.38}$$

and Lemma 1.3.2 is proven. □

Lemma 1.3.3. *Let $u(t)$ be a solution of equation (1.4) and let $t \geqslant T_0$, where T_0 is the same as in Lemma 1.3.1. Then, the following estimate holds:*

$$\|\partial_t u(t)\|_{-1}^2 + \|u(t)\|_{H^2(\Omega)}^2 + \int_t^{t+1} \|\partial_t u(s)\|_{H^1(\Omega)}^2 \mathrm{d}s \leqslant e^{K_1 t} Q(\|u_0\|_{H^2(\Omega)}),\ \forall t \geqslant T_0,$$

where K_1 is a positive constant and Q is some monotonic function.

Proof:
Differentiating (1.4) with respect to t and setting $\theta(t) = \partial_t u(t)$, we find:

$$\begin{cases} A_1^{-1}\partial_t \theta + A_2\theta + f'(u(t))\theta = 0, \\ \theta|_{\partial\Omega} = 0,\ \theta|_{t=T_0} = \partial_t u(T_0). \end{cases} \tag{1.39}$$

Multiplying (1.39) by $\theta(t)$, integrating over Ω and using the fact that $f'(u) \geqslant -\kappa$, we have:

$$\begin{aligned} \frac{d}{dt}\|\theta(t)\|_{-1}^2 + 2|A_2^{1/2}\theta(t)|^2 \leqslant & 2\kappa|\theta(t)|^2 \\ \leqslant & 2c\kappa\|\theta(t)\|_{-1}|A_2^{1/2}\theta(t)| \\ \leqslant & |A_2^{1/2}\theta(t)|^2 + c^2\kappa^2\|\theta(t)\|_{-1}^2, \end{aligned} \tag{1.40}$$

for an appropriate positive constant c. Applying Gronwall inequality to estimate (1.40), we obtain:

$$\|\theta(t)\|_{-1}^2 + \int_t^{t+1} |A_2^{1/2}\theta(s)|^2 \mathrm{d}s \leqslant e^{K_1 t}\|\theta(T_0)\|_{-1}^2,\ t \geqslant T_0, \tag{1.41}$$

for some positive constant K_1. Using Lemma 1.3.2, estimate (1.40) gives:

$$\|\partial_t u(t)\|_{-1}^2 + \int_t^{t+1} \|\partial_t u(s)\|_{H^1(\Omega)}^2 \mathrm{d}s \leqslant e^{K_1 t}(Q(\|u_0\|_{H^2(\Omega)})). \tag{1.42}$$

Interpreting the parabolic equation (1.4) as an elliptic boundary value problem:

$$-A_2 u(t) - f(u(t)) = h(t) := A_1^{-1}\partial_t u(t),\ u(t)|_{\partial\Omega} = 0, \tag{1.43}$$

for every fixed $t \geqslant T_0$, estimate (1.42) implies that:

$$|h(t)|^2 \leqslant \|\partial_t u(t)\|_{-1}^2 \leqslant e^{K_1 t}(Q(\|u_0\|_{H^2(\Omega)})). \tag{1.44}$$

We deduce from estimate (1.44) that:

$$\|u(t)\|_{H^2(\Omega)}^2 \leqslant c|A_2 u(t)|^2 \leqslant Ce^{K_1 t}(Q(\|u_0\|_{H^2(\Omega)})), \tag{1.45}$$

and the lemma is proven. □

Corollary 1.3.4. *Let the above assumptions hold and let $u(t)$ be a solution of equation (1.4). Then, the following estimate is valid, for every $t \geqslant 0$:*

$$\|u(t)\|_{H^2(\Omega)} \leqslant e^{Kt}(Q(\|u_0\|_{H^2(\Omega)})), \tag{1.46}$$

for some positive constant K and a monotonic function Q.

Proof:
Estimate (1.46) is an immediate consequence of Lemma 1.3.1 and Lemma 1.3.3. □

Theorem 1.3.5. *Let us take $u_0 \in W$. Then, there exists a unique solution u of problem (1.1) with initial datum u_0 such that:*

$$u \in L^\infty([0,T];H^2(\Omega)) \cap L^2([0,T];H^4(\Omega)) \text{ and } \partial_t u \in L^2([0,T];H^1(\Omega)). \quad (1.47)$$

Proof:
This theorem is an immediate consequence of Lemma 1.3.1 and 1.3.3. □

Remark 1.3.6. Theorem 1.3.5 is also true for every function f such that $f \in C^2(\mathbb{R})$, $f(0) = 0$ and $f'(s) \geqslant -\kappa$, where $\kappa \geqslant 0$.

1.4 Existence of the global attractor

In this section, we are interested in proving the existence of a global attractor for problem (1.1). We have the following:

Lemma 1.4.1. *Problem (1.1) generates the following semigroup on the phase space H:*

$$\begin{aligned} S(t) : H &\longrightarrow H \\ u_0 &\longmapsto S(t)u_0 = u(t),\ t \geqslant 0, \end{aligned}$$

where $u(t)$ is the unique solution of problem (1.1) with initial datum u_0 at time t. Furthermore, this semigroup is Lipschitz continuous in the $H^{-1}(\Omega)-$topology,

$$\|S(t)u_1 - S(t)u_2\|_{-1}^2 + \int_t^{t+1} \|S(s)u_1 - S(s)u_2\|_{H^1(\Omega)}^2 \mathrm{d}s \leqslant ce^{ct}\|u_1 - u_2\|_{-1}^2,$$

for any u_1, $u_2 \in H$, where c is a positive constant independent of t. Thus, $S(t)$ can be uniquely extended, by continuity, to a semigroup, still denoted by $S(t)$ acting on $H^{-1}(\Omega)$.

This Lemma is a direct consequence of (1.23), (1.24) and Theorem 1.2.1.
To prove the existence of a global attractor we show the existence of absorbing sets in $H^{-1}(\Omega)$, $L^2(\Omega)$, $H_0^1(\Omega)$ and in $H^2(\Omega)$.

An absorbing set in $H^{-1}(\Omega)$

Proposition 1.4.2. *Problem (1.1) has an absorbing set in $H^{-1}(\Omega)$. More precisely, there exists a constant ρ_0 and a time $t_0(\|u_0\|_{-1})$ such that, for the solution $u(t) = S(t)u_0$, we have:*

$$\|u(t)\|_{H^{-1}(\Omega)} \leqslant \rho_0 \text{ for all } t \geqslant t_0(||u_0||_{-1}).$$

Moreover,

$$\int_t^{t+r} |A_2^{1/2}u(s)|^2 ds + \int_t^{t+r} \int_\Omega F(u(s))\mathrm{d}x\ \mathrm{d}s \leqslant \rho_0^1(r) \text{ for all } t \geqslant t_0(\|u_0\|_{-1}),$$

where $\rho_H^1(r)$ is a positive constant depending on u_0 but independent of t.

Proof:
Multiplying equation (1.4) by u, we obtain:

$$\frac{1}{2}\frac{d}{dt}|A_1^{-1/2}u(t)|^2 + |A_2^{1/2}u(t)|^2 + (f(u(t)), u(t)) = 0.$$

Since

$$\begin{aligned}(f(u), u) \geqslant & \frac{a_{2p-1}}{2}\int_\Omega u^{2p}\mathrm{d}x - c_1|\Omega| \text{ (using (1.2))} \\ \geqslant & \frac{3}{4p}a_{2p-1}\int_\Omega u^{2p}\mathrm{d}x - c_1|\Omega| \\ \geqslant & \int_\Omega F(u)dx - (c_1+c_3)|\Omega| \text{ (using (1.3))},\end{aligned}$$

we find:

$$\frac{1}{2}\frac{d}{dt}|A_1^{-1/2}u|^2 + |A_2^{1/2}u|^2 + \int_\Omega F(u)\mathrm{d}x \leqslant k_0, \tag{1.48}$$

where $k_0 = (c_1+c_3)|\Omega|$. Using the fact that $|A_1^{-1/2}u|^2 \leqslant c'|A_2^{1/2}u|^2$ for all $u \in H_0^1(\Omega)$, we obtain:

$$\frac{1}{2}\frac{d}{dt}|A_1^{-1/2}u|^2 + \frac{1}{2c'}|A_1^{-1/2}u|^2 + \int_\Omega F(u)\mathrm{d}x \leqslant k_0.$$

Thus using (1.3), we have:

$$\frac{1}{2}\frac{d}{dt}|A_1^{-1/2}u|^2 + \frac{1}{2c'}|A_1^{-1/2}u|^2 + \frac{1}{4p}a_{2p-1}\int_\Omega u^{2p}\mathrm{d}x \leqslant k_1,$$

where $k_1 = k_0 + c_3|\Omega|$. The Gronwall's inequality leads to:

$$|A_1^{-1/2}u(t)|^2 \leqslant |A_1^{-1/2}u_0|^2\exp(-\frac{1}{c'}t) + 2c'k_1\left(1-\exp(-\frac{1}{c'}t)\right),\ \forall t>0.$$

It follows that if

$$t \geqslant t_0 = t_0(\|u_0\|_{-1}) = c_1|\Omega|\ln\left(\frac{\|u_0\|_{-1}^2}{2c'k_1}\right),$$

then:

$$\|u\|_{-1}^2 = |A_1^{-1/2}u|^2 \leqslant 4c'k_1 = \rho_0^2. \tag{1.49}$$

To deduce the integral bound on $|A_2^{1/2}u|$ and $\int_\Omega F(u)\mathrm{d}x$, we return to (1.48) and we integrate with respect to t from t to $t+r$, for $r>0$ a fixed constant. We find:

$$\begin{aligned}\int_t^{t+r}|A_2^{1/2}u(s)|^2\mathrm{d}s + \int_t^{t+r}\int_\Omega F(u(s))\mathrm{d}x\,\mathrm{d}s &\leqslant k_1 r + 1/2\|u(t)\|_{H^{-1}(\Omega)}^2 \\ &\leqslant k_1 r + \rho_0^2 := \rho_0^1(r),\end{aligned} \tag{1.50}$$

for all $t \geqslant t_0$. □

An absorbing set in $L^2(\Omega)$

Proposition 1.4.3. *Problem* (1.1) *has an absorbing set in* $L^2(\Omega)$. *More precisely, there exists a constant* ρ_H *and a time* $t_1(\|u_0\|_{-1})$ *such that, for the solution* $u(t) = S(t)u_0$, *we have:*

$$|u(t)| \leqslant \rho_H \text{ for all } t \geqslant t_1(||u_0||_{-1}).$$

Moreover,

$$\int_t^{t+r} |A_2 u(s)|^2 ds \leqslant \rho_H^1(r) \text{ for all } t \geqslant t_1(\|u_0\|_{-1}).$$

where $\rho_H^1(r)$ *is a positive constant depending on* u_0 *but independent of* t.

Proof:
Multiplying equation (1.1) by u and using (1.2), we have:

$$\frac{1}{2}\frac{d}{dt}|u|^2 + |A_2 u|^2 + |A_2^{1/2}u|^2 + \frac{a_{2p-1}}{2}\int_\Omega u^{2p}\mathrm{d}x \leqslant c_1|\Omega| + c|A_2^{1/2}u|^2. \tag{1.51}$$

Consequently, we obtain:

$$\frac{1}{2}\frac{d}{dt}|u|^2 \leqslant c_1|\Omega| + c|A_2^{1/2}u|^2.$$

In what follows, we use the technical result:

Lemma 1.4.4. *(The uniform Gronwall lemma). Let* g, h, y, *be three positive locally integrable functions on* $]t_0, +\infty[$ *such that* y' *is locally integrable on* $]t_0, +\infty[$, *and which satisfy:*

$$\frac{dy}{dt} \leqslant gy + h \quad \text{for } t \geqslant t_0,$$

$$\int_t^{t+r} g(s)\mathrm{d}s \leqslant a_1,\ \int_t^{t+r} h(s)\mathrm{d}s \leqslant a_2,\ \int_t^{t+r} y(s)\mathrm{d}s \leqslant a_3 \text{ for } t \geqslant t_0,$$

where r, a_1, a_2, a_3, *are positive constants. Then, we have:*

$$y(t+r) \leqslant (a_3/r + a_2)e^{a_1}, \quad \forall t \geqslant t_0.$$

Applying the uniform Gronwall lemma , the following estimate holds:

$$|u(t+r)|^2 \leqslant 2\left(\frac{c'(r)}{r} + c_1|\Omega| + c\rho_0^1(r)\right), \quad \forall t \geqslant t_0,$$

and

$$|u(t)|^2 \leqslant (\rho_H(r))^2 = \left(\frac{c'(r)}{r} + c_1|\Omega| + c\rho_0^1(r)\right), \quad \forall t \geqslant t_1 := t_0 + r. \tag{1.52}$$

Integrating (1.51) from t to $t+r$ and using (1.50), (1.52), we find:

$$\begin{aligned}\int_t^{t+r} |A_2 u|^2 ds \leqslant & c_1|\Omega|r + c\int_t^{t+r} |A_2^{1/2}u(s)|^2\mathrm{d}s + \frac{1}{2}|u(t)|^2 \\ \leqslant & c_1|\Omega|r + c\rho_0^1(r) + \frac{1}{2}(\rho_H(r))^2 := \rho_H^1(r),\end{aligned}$$

$\forall t \geqslant t_1$. This completes the proof of Proposition 1.4.2. □

An absorbing set in $H_0^1(\Omega)$

Proposition 1.4.5. *Equation* (1.1) *has an absorbing set in* $H_0^1(\Omega)$. *More precisely, there exists a constant* $\rho_V(r)$ *and a time* $t_1(\|u_0\|_{-1})$ *such that:*

$$||u(t)|| \leqslant \rho_V(r) \text{ for all } t \geqslant t_1(\|u_0\|_{-1}).$$

Proof:
We set

$$\mathcal{E}(u) = \frac{1}{2}|A_2^{1/2}u|^2 + \int_\Omega F(u)\mathrm{d}x \text{ and } K(u) = -\Delta u + f(u).$$

Multiplying (1.1) by $K(u)$, we can prove that we have:

$$\frac{d}{dt}\mathcal{E}(u) + |A_2^{1/2}K(u)|^2 + |K(u)|^2 = 0,$$

which shows that:

$$\mathcal{E}(u(t)) \leqslant \mathcal{E}(u_0), \quad \forall t \geqslant 0.$$

Equation (1.48) can be written as follow:

$$\frac{1}{2}\frac{d}{dt}|A_1^{-1/2}u|^2 + \mathcal{E}(u) \leqslant k_0. \tag{1.53}$$

Integrating (1.53) with respect to t from t to $t+r$, we obtain:

$$\frac{1}{2}|A_1^{-1/2}u(t+r)|^2 + \int_t^{t+r} \mathcal{E}(u(s))\mathrm{d}s \leqslant k_0 r + \frac{1}{2}|A_1^{-1/2}u(t)|^2. \tag{1.54}$$

Using the fact that $\mathcal{E}$ decays along the orbits and that $|A_1^{-1/2}u(t)| \leqslant \rho_0,\ \forall t \geqslant t_0$, we deduce:

$$\mathcal{E}(u(t+r)) \leqslant k_0 + \frac{1}{2r}\rho_0^2, \quad \forall t \geqslant t_0, \tag{1.55}$$

and

$$|A_2^{1/2}u|^2 + \frac{1}{2p}a_{2p-1}\int_\Omega u^{2p}\mathrm{d}x \leqslant \rho_V^2(r) := 2k_0 + 2c_3|\Omega| + \frac{1}{r}\rho_0^2, \quad \forall t \geqslant t_1 := t_0 + r. \tag{1.56}$$

□

An absorbing set in $H^2(\Omega)$ We can also prove the existence of an absorbing set in $H^2(\Omega)$. We actually prove the following result:

Proposition 1.4.6. *Problem* (1.1) *has an absorbing set in* $H^2(\Omega)$. *More precisely, there exists a constant* ρ *and a time* $t_2(\|u_0\|_{-1})$ *such that*

$$|A_2 u(t)| \leqslant \rho(r) \text{ for all } t \geqslant t_2(\|u_0\|_{-1}). \tag{1.57}$$

Proof:
Multiplying (1.4) by $\partial_t u$, we obtain:

$$\frac{1}{2}\frac{d}{dt}|A_2^{1/2}u|^2 + \frac{d}{dt}\int_\Omega F(u)\mathrm{d}x + \|\partial_t u\|_{-1}^2 = 0. \tag{1.58}$$

Using (1.56), we find:

$$\int_t^{t+r} \|\partial_t u(s)\|_{-1}^2 \mathrm{d}s \leqslant \rho_V'(r), \quad \forall t \geqslant t_1. \tag{1.59}$$

Differentiating (1.4) with respect to t, we have:

$$A_1^{-1}\partial_t\partial_t u + A_2\partial_t u + f'(u)\partial_t u = 0. \tag{1.60}$$

Multiplying (1.58) by $\partial_t u$ and using the fact that $f'(u) \geqslant -\kappa$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u\|_{-1}^2 + |A_2^{1/2}\partial_t u|^2 \leqslant \kappa|\partial_t u|^2 \leqslant c\kappa\|\partial_t u\|_{-1}^2|A_2^{1/2}\partial_t u|^2, \tag{1.61}$$

which yields:

$$\frac{d}{dt}\|\partial_t u\|_{-1}^2 \leqslant c\|\partial_t u\|_{-1}^2, \tag{1.62}$$

for some positive constant c. Using (1.59), (1.61) and the uniform Gronwall lemma, we deduce that:

$$\|\partial_t u\|_{-1}^2 \leqslant c(r), \quad \forall t \geqslant t_1 + r. \tag{1.63}$$

We rewrite (1.4) in the following form:

$$A_2 u + f(u) = -A_1^{-1}\frac{du}{dt}. \tag{1.64}$$

Multiplying (1.64) by $A_2 u$ and using that $f'(u) \geqslant -\kappa$, we obtain:

$$\begin{aligned} |A_2 u|^2 \leqslant& \kappa|A_2^{1/2}u|^2 + (A_1^{-1}\partial_t u, A_2 u) \\ \leqslant& \kappa|A_2^{1/2}u|^2 + \|\partial_t u\|_{-1}|A_2 u|, \end{aligned} \tag{1.65}$$

which yields:

$$|A_2 u|^2 \leqslant 2\kappa|A_2^{1/2}u|^2 + \|\partial_t u\|_{-1}^2. \tag{1.66}$$

Using (1.56), (1.63), we deduce from (1.66) that:

$$|A_2 u| \leqslant \rho(r), \quad \forall t \geqslant t_2 := t_1 + r,$$

for some positive constant $\rho(r)$ which depend on r and Proposition 1.4.6 is proven. □

Using the existence of an absorbing set in $H^2(\Omega)$, we can deduce the existence of the global attractor.
The following theorem gives the existence of the global attractor $\mathcal{A}$ in $H^{-1}(\Omega)$ for the semigroup $S(t)$. We recall that, by definition, a set $\mathcal{A} \subset H^{-1}(\Omega)$ is the global attractor for the semigroup $S(t)$ if the following properties are satisfied:

1. It is a compact subset of $H^{-1}(\Omega)$;

2. It is strictly invariant, i.e., $S(t)\mathcal{A} = \mathcal{A},\ \forall t > 0$;

3. It attracts all bounded sets in $H^{-1}(\Omega)$ as $t \to \infty$, i.e., for every bounded set $X \subset H^{-1}(\Omega)$ there exists a neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in $H^{-1}(\Omega)$ and a time $T = T(\mathcal{O})$ such that:
$$S(t)X \subset \mathcal{O}(\mathcal{A}),\ \ t \geqslant T.$$

Theorem 1.4.7. *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ in $H^{-1}(\Omega)$ which is bounded in $H^2(\Omega)$.*

Proof:
Let B_1 be a bounded set in $H^{-1}(\Omega)$ defined by
$$\|\varphi\|_{-1} \leqslant \rho_0,$$
with ρ_0 as in (1.49) and B_2 be a bounded set in $H^2(\Omega)$ defined by
$$|A_2\varphi| \leqslant \rho,$$
with ρ as in (1.57). B_2 is a bounded absorbing set in $H^2(\Omega)$, compact in $H^{-1}(\Omega)$, which implies the existence of the global attractor in $H^{-1}(\Omega)$. □

1.5 Existence of an exponential attractor

In this section, we prove the existence of an exponential attractor which by definition, contains the global attractor and has finite fractal dimension. To do this, we first recall the definition of the exponential attractor where $\mathcal{A}$ is the global attractor for the semigroup $\{S(t)\}_{t\geqslant 0}$:

Definition 1.5.1. Let X be a compact connected subset of a Banach space E. A compact set $\mathcal{M}$ is called an exponential attractor for the semigroup $\{S(t)\}_{t\geqslant 0}$ if $\mathcal{A} \subset \mathcal{M} \subset X$ and

1. $S(t)\mathcal{M} \subset \mathcal{M},\ \forall t \geqslant 0$.

2. $\mathcal{M}$ has finite fractal dimension, $d_F(\mathcal{M}) < \infty$.

3. There exist positive constants c_0 and c_1 such that for every $u_0 \in X$, we have:
$$dist_E(S(t)u_0, \mathcal{M}) \leqslant c_0 e^{-c_1 t},\ \forall t \geqslant 0, \tag{1.67}$$
where the pseudo-distance $dist$ is the standard Hausdorff pseudo-distance between two sets, defined by $dist_E(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_E$.

To prove the existence of an exponential attractor, we apply the following theorem (see [44]):

Theorem 1.5.2. *Let E and E_1 be two Hilbert spaces such that E_1 is compactly embedded into E and $S(t) : X \to X$ be a semigroup acting on a closed subset X of E. We assume that:*

1. $\forall x_1,\ x_2 \in X,\ \forall t > 0,$

$$\|S(t)x_1 - S(t)x_2\|_{E_1} \leqslant h(t)\|x_1 - x_2\|_E,$$

where the function h is continuous;

2. $(t, x) \mapsto S(t)x$ *is uniformly Hölder continuous on* $[0, T] \times B,\ \forall T > 0,\ \forall B \subset X$ *bounded.*

Then, $S(t)$ possesses an exponential attractor on X.

In order to apply this result to the semigroup $S(t)$ associated with problem (1.1), we set $E = H^{-1}(\Omega),\ E_1 = L^2(\Omega)$ and we consider the set

$$X = \overline{\bigcup_{t \geqslant\ t_0+2r} S(t)B_2},$$

where $t_0 + 2r$ is such that for all $t \geqslant t_0 + 2r$, we have $S(t)B_2 \subset B_2$ (X is thus compact in $H^{-1}(\Omega)$, bounded in $H^2(\Omega)$ and positively invariant by $S(t)$).
Let u_1 and u_2 be respectively two solutions of (1.1) with initial data in X. We set $w = u_1 - u_2$. Then, w verifies:

$$\begin{cases} A_1^{-1}\dfrac{dw}{dt} + A_2 w + l(t)w = 0, \\ w(0) = u_1(0) - u_2(0), \end{cases} \tag{1.68}$$

where $l(t) = \displaystyle\int_0^1 f'(su_1 + (1-s)u_2)ds$.
In order to complete the proof, we need to prove the following lemma:

Lemma 1.5.3. *Let $u_1(t)$ and $u_2(t)$ be two solutions of (1.1) such that $|A_2 u_i(0)| \leqslant \rho,\ i = 1, 2$. Then, the following estimate is valid:*

$$\|u_1(t) - u_2(t)\|^2_{H^{-1}(\Omega)} + \int_t^{t+1} \|w(s)\|^2_{H^1(\Omega)} ds \leqslant C_\rho e^{\alpha_\rho t}\|u_1(0) - u_2(0)\|^2_{H^{-1}(\Omega)}, \tag{1.69}$$

where the positive constants C_ρ and α_ρ depend on ρ.

Proof:
We have:

$$\|l(t)\|_{L^\infty(\Omega)} \leqslant C(\|u_1(0)\|_{H^2(\Omega)}, \|u_2(0)\|_{H^2(\Omega)}) \leqslant Q_\rho, \tag{1.70}$$

where the constant Q_ρ depends on the constant ρ. Multiplying (1.68) by w, integrating over Ω and using (1.70), we obtain the following inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{-1}(\Omega)} + |A_2^{1/2}w|^2 &= -\,(l(t)w, w) \\ &\leqslant \|l(t)\|_{L^\infty(\Omega)}\|w\|^2_{L^2(\Omega)} \\ &\leqslant cQ_\rho\|w\|_{H^{-1}(\Omega)}\|A_2^{1/2}w\|_{L^2(\Omega)}. \end{aligned} \tag{1.71}$$

Applying Gronwall's inequality to (1.71), we conclude the proof of the lemma.
The next lemma gives the $H^{-1}(\Omega) \to L^2(\Omega)$-smoothing property for the difference of two solutions. □

Lemma 1.5.4. *Let $u_1(t)$ and $u_2(t)$ be two solutions of (1.1) such that $|A_2 u_i(0)| \leqslant \rho$, $i = 1, 2$. Then, the following estimate is valid:*

$$t\|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} \leqslant R_\rho e^{\alpha_\rho t}\|u_1(0) - u_2(0)\|^2_{H^{-1}(\Omega)},\ t > 0, \tag{1.72}$$

where R_ρ is a positive constant depending on ρ.

Proof:
Multiplying (1.72) by $tA_2 w$ and integrating over Ω, we find:

$$\frac{1}{2}t\frac{d}{dt}(A_2 A_1^{-1} w, w) + t|A_2 w|^2 + t\left(l(t)w, A_2 w\right) = 0. \tag{1.73}$$

Setting $(A_2 A_1^{-1} w, w) = \|w\|^2_*$, where $\|.\|_* \sim |.|_{L^2(\Omega)}$, equation (1.73) yields:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(t\|w\|^2_*\right) + t|A_2 w|^2 \leqslant & \frac{1}{2}\|w\|^2_* - t\left(l(t)w, A_2 w\right) \\
\leqslant & \frac{1}{2}\|w\|^2_* + t\|l(t)\|_{L^\infty(\Omega)}|w||A_2 w| \\
\leqslant & \frac{1}{2}\|w\|^2_* + tQ_\rho|w||A_2 w| \\
\leqslant & \frac{1}{2}\|w\|^2_* + \frac{t}{2}Q^2_\rho|w|^2 + \frac{t}{2}|A_2 w|^2 \\
\leqslant & \frac{1}{2}\|w\|^2_* + \frac{t}{2}Q^2_\rho|A_2^{1/2} w|^2 + \frac{t}{2}|A_2 w|^2
\end{aligned}$$

Integrating with respect to t from 0 to t, we obtain:

$$\begin{aligned}
t\|w(t)\|^2_* + \int_0^t s|A_2 w(s)|^2 ds \leqslant & \int_0^t \|w\|^2_* ds + Q^2_\rho \int_0^t s|A_2^{1/2} w(s)|^2 ds \\
\leqslant & c\int_0^t |A_2^{1/2} w(s)|^2 ds + Q^2_\rho \int_0^t s|A_2^{1/2} w(s)|^2 ds \\
\leqslant & R_\rho e^{\alpha_\rho t}\|u_1(0) - u_2(0)\|^2_{H^{-1}(\Omega)},
\end{aligned}$$

where we use the fact that $\|w\|_* \leqslant c|A_2^{1/2} w|$ and Lemma 1.5.3.
Finally, we deduce:

$$t\|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} \leqslant R_\rho e^{\alpha_\rho t}\|u_1(0) - u_2(0)\|^2_{H^{-1}(\Omega)}.$$

□

Thus, the first condition of Theorem 1.5.2 is proven and it is sufficient to prove the second condition in order to prove the existence of an exponential attractor.

Lemma 1.5.5. *The semigroup $S(t)$ is uniformly Hölder continuous on $[0, T] \times B_2$ in the topology of $H^{-1}(\Omega)$, where $B_2 = \left\{u \in W, \|u\|_{H^2(\Omega)} \leqslant \rho\right\}$, i.e.*

$$\|S(t_1)u_{01} - S(t_2)u_{02}\|_{-1} \leqslant C_{\rho,T}\left(\|u_{01} - u_{02}\|_{H^{-1}(\Omega)} + |t_1 - t_2|^{1/2}\right),$$

where $u_{0i} \in B_2$, $t_i \leqslant T, i = 1, 2$, and $C_{\rho,T}$ is a positive constant depending on ρ and T.

Proof:
The Lipschitz continuity with respect to the initial conditions is an immediate corollary of lemma 1.5.3. In order to verify the Hölder continuity with respect to t, we multiply equation (1.4) by $\partial_t u$ and we integrate over Ω. We obtain:

$$|A_1^{-1/2}\partial_t u|^2 + \frac{1}{2}\frac{d}{dt}|A_2^{1/2}u|^2 + \frac{d}{dt}\int_\Omega F(u)dx = 0.$$

Integrating with respect to t from t_1 to t_2, we find:

$$\begin{aligned}\int_{t_1}^{t_2} |A_1^{-1/2}\partial_t u|^2 ds + \frac{1}{2}|A_2^{1/2}u(t_2)|^2 + \int_\Omega F(u(t_2))\mathrm{d}x \\ = \frac{1}{2}|A_2^{1/2}u(t_1)|^2 + \int_\Omega F(u(t_1))\mathrm{d}x.\end{aligned}$$

Using (1.3), we have:

$$\begin{aligned}\int_{t_1}^{t_2} |A_1^{-1/2}\frac{\partial u}{\partial t}|^2 ds + \frac{1}{2}|A_2^{1/2}u(t_2)|^2 + \frac{a_{2p-1}}{4p}\int_\Omega u^{2p}(t_2)\mathrm{d}x \\ \leqslant 2c_3|\Omega| + \frac{1}{2}|A_2^{1/2}u(t_1)|^2 + \frac{3a_{2p-1}}{4p}\int_\Omega u^{2p}(t_1)\mathrm{d}x \\ \leqslant C(\rho).\end{aligned}$$

Consequently, we deduce:

$$\begin{aligned}\|u(t_1) - u(t_2)\|_{H^{-1}(\Omega)} &= \left\|\int_{t_1}^{t_2}\partial_t u(s)ds\right\|_{H^{-1}(\Omega)} \leqslant \int_{t_1}^{t_2}\|\partial_t u(s)\|_{H^{-1}(\Omega)}\mathrm{d}s \\ &\leqslant |t_1 - t_2|^{1/2}\left(\int_{t_1}^{t_2}\|\partial_t u(s)\|^2_{H^{-1}(\Omega)}\mathrm{d}s\right)^{1/2} \\ &\leqslant C_{\rho,T}|t_1 - t_2|^{1/2},\end{aligned}$$

where $C_{\rho,T}$ is a positive constant depending on ρ and T, which yields the Hölder continuity with respect to t.

□

Hence, we deduce the existence of an exponential attractor for problem (1.1).

Remark 1.5.6. We recall that an exponential attractor always contains the global attractor. Consequently, we deduce that the global attractor has finite fractal dimension in $H^{-1}(\Omega)$.

Remark 1.5.7. For the problem with Neumann boundary conditions, we set $g(s) = f(s) - s$, thus we obtain the following reformulation:

$$A_1^{-1}\partial_t u + A_1 u + g(u) = 0.$$

Noting that g satisfies the same properties as f, we proceed as above and we obtain the same previous results.

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Chapitre 2

Well-Posedness and Long Time Behavior of a Perturbed Cahn-Hilliard System with Regular Potentials

Sur le caractère bien posé et le comportement asymptotique d'une équation de type Cahn-Hilliard perturbée

Ce chapitre est constitué de l'article **Well-posedness and long time behavior of a perturbed Cahn-Hilliard system with regular potentials**, écrit en collaboration avec Alain Miranville et Madalina Petcu, article paru en 2013 dans dans *Journal of Asymptotic Analysis*, volume 84, pages 147-179.

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Well-posedness and long time behavior of a perturbed Cahn-Hilliard system with regular potentials

Haydi Israel[1], Alain Miranville[1] and Madalina Petcu[1,2]

[1]UMR 7348 CNRS. Laboratoire de Mathématiques et Applications - Université de Poitiers - SP2MI - Boulevard Marie et Pierre Curie, Téléport 2
BP30179 - 86962 Futuroscope Chasseneuil Cedex - FRANCE.

[2]Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Abstract. The aim of this paper is to study the well-posedness and long time behavior, in terms of finite-dimensional attractors, of a perturbed Cahn-Hilliard equation. This equation differs from the usual Cahn-Hilliard by the presence of the term $\varepsilon(-\Delta u + f(u))$. In particular, we prove the existence of a robust family of exponential attractors as ε goes to zero.

2.1 Introduction

We consider the following boundary value problem in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$:

$$\begin{cases} \partial_t u + \Delta^2 u - \Delta f(u) - \varepsilon\Delta u + \varepsilon f(u) = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \\ u|_{t=0} = u_0, \end{cases} \tag{2.1}$$

where f is the derivative of a nonconvex potential and the unknown u is the relative concentration of one phase. We assume throughout this paper that $n \leqslant 3$.
When $\varepsilon = 0$, we recover the well-known Cahn-Hilliard equation (see [38], [47], [54]) and when $\varepsilon > 0$, equation (2.1) may be viewed as a combination of the well-known Cahn-Hilliard equation and the Allen-Cahn equation (see [30], [31], [32], [37]).
We recall that the Cahn-Hilliard equation describes the behavior of two-phase systems, in particular in spinodal decomposition, i.e. in the case of rapid separation of phases when the material is cooled down sufficiently and, when $\varepsilon > 0$, the equation describes a simplified model of adsorption to and desorption from the surface.
Such a model has been studied in [27], [28] and [31], for a smooth nonlinearity in [28], [31] and for a singular nonlinearity in [27] where questions such as existence

and uniqueness of solutions, existence of the global attractor and an exponential attractor have been addressed.
We rewrite problem (2.1) as a system of second order equations:

$$\begin{cases} \partial_t u = \Delta w - \varepsilon w & \text{in } \Omega, \\ w = -\Delta u + f(u), & \\ u = \Delta u = 0 & \text{on } \partial\Omega, \\ u|_{t=0} = u_0. & \end{cases} \tag{2.2}$$

We denote by F the antiderivative of f vanishing at 0 and assume that

$$\begin{cases} f \in C^2(\mathbb{R}), \\ f(0) = 0, \\ f'(s) \geqslant -\kappa, \\ f(s)s \geqslant p b_{2p} s^{2p} - c \geqslant c(s^2 + s^{2p}) - c, \\ |f(s)| \leqslant c(|s|^{2p-1} + 1), \\ \frac{1}{2} b_{2p} s^{2p} - c \leqslant F(s) \leqslant \frac{3}{2} b_{2p} s^{2p} + c, \end{cases} \tag{2.3}$$

for all $s \in \mathbb{R}$ and where κ, c and b_{2p} are positive constants and $p \geqslant 2$ is an integer. Typical choice for f is

$$f(s) = s^3 - s.$$

This paper is organized as follows. In Section 2, we give some useful assumptions and notation. Then, in Section 3, we derive uniform a priori estimates for approximated solutions which allow us to pass to the limit in the approximated problem to study the well-posedness, namely, the existence and uniqueness of a weak solution as stated in Theorem 2.3.1. Section 4 is dedicated to the proof of some additional regularity for the solution. It follows from the well-posedness result that the system generates a continuous semigroup in a suitable phase space, which allows to study the existence of the global attractor in Section 5. In Section 6, we prove that the fractal dimension of the global attractor is finite by studying the existence of exponential attractors. Finally, Section 7 is devoted to the proof of the continuity of exponential attractors for the perturbed system (2.1) and to the derivation of the corresponding estimate for the symmetric distance.

Remark 2.1.1. Neumann boundary conditions, namely, $\partial_n u = \partial_n \Delta u = 0$ on $\partial\Omega$, are also relevant in the context of the Allen-Cahn and Cahn-Hilliard equations. In that case, the limit problem, i.e., the Cahn-Hilliard equation, is a conservation law, in the sense that the spatial average of the order parameter u is a conserved quantity. This brings additional difficulties in the study of the continuity of exponential attractors and will be studied elsewhere.

2.2 Notation and assumptions

We introduce the following spaces:

$$H = L^2(\Omega), V = H_0^1(\Omega), W = H^2(\Omega) \cap H_0^1(\Omega).$$

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the usual norm and inner product in H.
Let

$$A_\varepsilon = -\Delta + \varepsilon I : D(A_\varepsilon) \subset H \to H$$

with $D(A_\varepsilon)= W$. The operator A_ε is a strictly positive self-adjoint linear operator with compact inverse A_ε^{-1}. Then, problem (2.2) can be reformulated as follows:

$$\begin{cases} A_\varepsilon^{-1}\partial_t u - \Delta u + f(u) = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u|_{t=0} = u_0. \end{cases} \tag{2.4}$$

For $0 \leqslant \varepsilon \leqslant 1$ and $u \in D(A_\varepsilon)$, we have the following:

$$\|\nabla u\| \leqslant \|A_\varepsilon^{1/2} u\| \leqslant \|u\|_{H^1(\Omega)} \Longrightarrow \|A_\varepsilon^{1/2} u\| \sim \|u\|_{H^1(\Omega)}, \tag{2.5}$$

$$\|\Delta u\| \leqslant \|A_\varepsilon u\| \leqslant c\|u\|_{H^2(\Omega)} \Longrightarrow \|A_\varepsilon u\| \sim \|u\|_{H^2(\Omega)}, \tag{2.6}$$

$$\|A_\varepsilon^{-1/2} u\| \leqslant c\|A_\varepsilon^{1/2} u\|, \tag{2.7}$$

$$\|u\|_{-1} \sim \|A_\varepsilon^{-1/2} u\|, \tag{2.8}$$

$$\|u\|_{-2} \sim \|A_\varepsilon^{-1} u\|, \tag{2.9}$$

$$(A_\varepsilon(-\Delta)^{-1} u, u) \sim \|u\|^2, \tag{2.10}$$

and

$$((-\Delta)A_\varepsilon^{-1} u, u) \sim \|u\|^2, \tag{2.11}$$

where c is independent of ε, $\|v\|_{-1}^2 = ((-\Delta)^{-1} v, v)$ and all the equivalences are independent of ε. Indeed, we have, if u is regular enough:

$$\|\nabla u\|^2 \leqslant \|A_\varepsilon^{1/2} u\|^2 = (A_\varepsilon u, u) = \|\nabla u\|^2 + \varepsilon\|u\|^2 \leqslant \|\nabla u\|^2 + \|u\|^2 = \|u\|_{H^1(\Omega)}^2,$$
$$\|\Delta u\|^2 \leqslant \|A_\varepsilon u\|^2 = (A_\varepsilon u, A_\varepsilon u) = \|\Delta u\|^2 + 2\varepsilon\|\nabla u\|^2 + \varepsilon^2\|u\|^2 \leqslant c\|u\|_{H^2(\Omega)}^2.$$

We then use Poincaré's inequality since $u = 0$ on $\partial\Omega$ and we deduce (2.5) and (2.6). Now, using the inclusions $V \subset H \subset V'$, the scalar product:

$$a(u, v) = (\nabla u, \nabla v), \ \forall u, v \in V$$

defines a linear operator $A : D(A) = W \to H$. The operator A is the Laplace operator with Dirichlet boundary conditions and is a nonnegative self-adjoint operator; it has an orthonormal basis of eigenvectors $\{e_j\}_j$ associated to the eigenvalues $\{\lambda_j\}_j$, with

$$0 < \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots, \ \lambda_j \to +\infty \text{ as } j \to +\infty.$$

The family $\{e_j\}_j$ may be assumed to be normalized in the norm of H, i.e.,

$$(e_i, e_j) = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We also note that the operator A_ε has the same orthonormal basis of eigenvectors $\{e_j\}_j$ associated to the eigenvalues $\{\lambda_{\varepsilon,j}\}_j$, with $\lambda_{\varepsilon,j} = \lambda_j + \varepsilon$. We have

$$\lambda_j \leqslant \lambda_{\varepsilon,j} \leqslant \lambda_j + 1, \ \forall j \in \mathbb{N},$$

hence (2.7). We note that

$$A_\varepsilon(-\Delta)^{-1}u = (-\Delta + \varepsilon I)(-\Delta)^{-1}u = u + \varepsilon(-\Delta)^{-1}u,$$

hence

$$(-\Delta)^{-1}u = A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u.$$

Therefore,

$$\|u\|_{-1}^2 = ((-\Delta)^{-1}u, u) = (A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u, u) = \|A_\varepsilon^{-1/2}u\|^2 + \varepsilon(A_\varepsilon^{-1}(-\Delta)^{-1}u, u).$$

Using the fact that A_ε^{-1} and $(-\Delta)^{-1}$ are nonnegative self-adjoint operators which commute, we obtain

$$\varepsilon(A_\varepsilon^{-1}(-\Delta)^{-1}u, u) \geqslant 0,$$

which yields

$$\|A_\varepsilon^{-1/2}u\|^2 \leqslant \|u\|_{-1}^2. \tag{2.12}$$

We also have $A_\varepsilon^{-1/2}(-\Delta)^{-1/2} = (-\Delta)^{-1/2}A_\varepsilon^{-1/2}$, hence

$$\begin{aligned} \|u\|_{-1}^2 = ((-\Delta)^{-1}u, u) =& (A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u, u) \\ =& \|A_\varepsilon^{-1/2}u\|^2 + \varepsilon\|A_\varepsilon^{-1/2}(-\Delta)^{-1/2}u\|^2 \\ \leqslant& \|A_\varepsilon^{-1/2}u\|^2 + \|(-\Delta)^{-1/2}A_\varepsilon^{-1/2}u\|^2 \\ \leqslant& \|A_\varepsilon^{-1/2}u\|^2 + c'\|A_\varepsilon^{-1/2}u\|^2, \end{aligned}$$

which yields

$$\|u\|_{-1}^2 \leqslant c\|A_\varepsilon^{-1/2}u\|^2. \tag{2.13}$$

From estimates (2.12) and (2.13), we deduce (2.8).
We further have

$$\begin{aligned} \|u\|_{-2}^2 =& ((-\Delta)^{-1}u, (-\Delta)^{-1}u) \\ =& (A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u, A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u) \\ =& \|A_\varepsilon^{-1}u\|^2 + \varepsilon^2\|A_\varepsilon^{-1}(-\Delta)^{-1}u\|^2 + 2\varepsilon(A_\varepsilon^{-1}(-\Delta)^{-1}u, A_\varepsilon^{-1}u) \\ =& \|A_\varepsilon^{-1}u\|^2 + \varepsilon^2\|A_\varepsilon^{-1}(-\Delta)^{-1}u\|^2 + 2\varepsilon\|(-\Delta)^{-1/2}A_\varepsilon^{-1}u\|^2 \\ \leqslant& \|A_\varepsilon^{-1}u\|^2 + \|(-\Delta)^{-1}A_\varepsilon^{-1}u\|^2 + 2\|(-\Delta)^{-1/2}A_\varepsilon^{-1}u\|^2 \\ \leqslant& c\|A_\varepsilon^{-1}u\|^2, \end{aligned}$$

and, since $\varepsilon^2\|A_\varepsilon^{-1}(-\Delta)^{-1}u\|^2 + 2\varepsilon(A_\varepsilon^{-1}(-\Delta)^{-1}u, A_\varepsilon^{-1}u) \geqslant 0$, we find

$$\|A_\varepsilon^{-1}u\|^2 \leqslant \|u\|_{-2}^2.$$

Then, we deduce that

$$\|A_\varepsilon^{-1}u\|^2 \leqslant \|u\|_{-2}^2 \leqslant c\|A_\varepsilon^{-1}u\|^2,$$

where c is independent of ε.
Now,

$$\begin{aligned}(A_\varepsilon(-\Delta)^{-1}u,u) = ((-\Delta+\varepsilon I)(-\Delta)^{-1}u,u) =&(u+\varepsilon(-\Delta)^{-1}u,u)\\ =&\|u\|^2+\varepsilon\|u\|_{-1}^2\\ \leqslant&\|u\|^2+\|u\|_{-1}^2\\ \leqslant&c\|u\|^2\end{aligned}$$

and

$$\|u\|^2 \leqslant (A_\varepsilon(-\Delta)^{-1}u,u),$$

so that

$$(A_\varepsilon(-\Delta)^{-1}u,u) \sim \|u\|^2.$$

We also have

$$\begin{aligned}((-\Delta)A_\varepsilon^{-1}u,u) =&((-\Delta)A_\varepsilon^{-1}u, A_\varepsilon(-\Delta)^{-1}(-\Delta)A_\varepsilon^{-1}u)\\ \leqslant&c\|(-\Delta)A_\varepsilon^{-1}u\|^2(\text{using (2.10)})\\ \leqslant&c\|A_\varepsilon A_\varepsilon^{-1}u\|^2(\text{using (2.6)})\\ \leqslant&c\|u\|^2\end{aligned}$$

and

$$\begin{aligned}\|u\|^2 =&\|A_\varepsilon^{1/2}(-\Delta)^{-1/2}(-\Delta)^{1/2}A_\varepsilon^{-1/2}u\|^2\\ \leqslant&c\|(-\Delta)^{1/2}A_\varepsilon^{-1/2}u\|^2\\ \leqslant&c((-\Delta)A_\varepsilon^{-1}u,u),\end{aligned}$$

which yields

$$((-\Delta)A_\varepsilon^{-1}u,u) \sim \|u\|^2.$$

The variational formulation of problem (2.4) reads
Find $u:[0,T]\to V$ *such that*

$$\begin{cases}(A_\varepsilon^{-1}\partial_t u,q)+(\nabla u,\nabla q)+(f(u),q)=0,\ \forall q\in V,\\ u(0)=u_0,\end{cases}\tag{2.14}$$

$\forall T>0$.
In what follows, unless mentioned explicitly, the same letter Q denotes monotone increasing functions and the same letter c denotes positive constants independent of ε, possibly changing at different occurrences.

2.3 Uniform a priori estimates. Existence and uniqueness of solutions.

In order to prove the existence of solutions for problem (2.4), we use a Galerkin scheme. We first define an m-dimensional system which approximates the initial

problem. The resolution of the system proves the existence of an approximated solution. We then obtain a priori estimates which allow us to justify the passage to the limit in the approximated problem and obtain a solution to the initial problem. We have the following theorem.

Theorem 2.3.1. *Let us take $u_0 \in H$. Then, there exists a unique solution u of problem* (2.4) *with initial datum u_0 such that*

$$u \in L^\infty([0,T];H) \cap L^2([0,T];W) \cap L^{2p}(0,T;L^{2p}(\Omega)).$$

Furthermore, if $u_0 \in V \cap L^{2p}(\Omega)$, then

$$u \in L^\infty([0,T];V \cap L^{2p}(\Omega)) \cap L^2([0,T];W) \text{ and } \partial_t u \in L^2([0,T];H^{-1}(\Omega)).$$

Uniqueness of the solution: Let u and v be two solutions of (2.4) on the time interval $[0,T]$. We set $w = u - v$, then w verifies the following equation:

$$\begin{cases} A_\varepsilon^{-1}\partial_t w + \nabla w + l(t)w = 0, \\ w(0) = u(0) - v(0), \end{cases} \tag{2.15}$$

where $l(t) = \displaystyle\int_0^1 f'\left(su(t) + (1-s)v(t)\right) \mathrm{d}s$. Multiplying (2.15) by w and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{-1/2}w\|^2 + \|\nabla w\|^2 + (l(t)w, w) = 0.$$

Using the fact that $f'(s) \geqslant -\kappa$, we have

$$(l(t)w, w) \geqslant -\kappa\|w\|^2.$$

Using the following interpolation inequality:

$$\|w\|^2_{L^2(\Omega)} \leqslant c\|w\|_{-1}\|\nabla w\| \leqslant c\|A_\varepsilon^{-1/2}w\|\|\nabla w\|,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{-1/2}w\|^2 + \frac{1}{2}\|\nabla w\|^2 \leqslant \frac{c^2\kappa^2}{2}\|A_\varepsilon^{-1/2}w\|^2. \tag{2.16}$$

Applying Gronwall's lemma to (2.16), we find

$$\|A_\varepsilon^{-1/2}w(t)\|^2 + \int_0^t \|\nabla w(s)\|^2 \mathrm{d}s \leqslant \|A_\varepsilon^{-1/2}(u(0) - v(0))\|^2 e^{c^2\kappa^2 t}. \tag{2.17}$$

Relation (2.17) shows the continuous dependence of the solution on the initial data in the $H^{-1}-$norm and, in particular, when $u(0) = v(0)$, it implies the uniqueness of the solution.

Existence of a local solution: We denote by E_m the space

$$E_m = \text{span}\{e_1, e_2, \dots, e_m\}$$

and by P_m the orthogonal projection from V onto E_m:

$$P_m h = \sum_{j=1}^{m} (h, e_j)\, e_j,$$

where $\{e_j\}_j$ are the eigenfunctions of the operator A.
For any $m \in \mathbb{N}$, we look for functions of the form

$$u_m(t) = \sum_{j=1}^{m} u_{m_j}(t)\, e_j,$$

solving the approximated problem

$$\begin{cases} A_\varepsilon^{-1}\partial_t u_m - \Delta u_m + f(u_m) = 0, \\ u_m(0) = P_m u_0. \end{cases} \tag{2.18}$$

The variational formulation of problem (2.18) reads
Find $u_m : [0,T] \to E_m$ *such that*

$$\begin{cases} \dfrac{d}{dt}\left(A_\varepsilon^{-1}\displaystyle\sum_{j=1}^{m} u_{m_j}(t)\, e_j, e_i\right) + \left(\displaystyle\sum_{j=1}^{m} u_{m_j}(t)\, \nabla e_j, \nabla e_i\right) + (f(u_m), e_i) = 0,\ i = 1, ..., m, \\ u_m(0) = P_m u_0. \end{cases} \tag{2.19}$$

We rewrite problem (2.19) as follows:

$$M_\varepsilon \frac{dY}{dt} + MY + N(Y) = 0,$$

where $M_\varepsilon = \left((A_\varepsilon^{-1}(e_i), e_j) \right)_{i,j=1,\dots,m}$, $M = \left((\nabla e_i, \nabla e_j) \right)_{i,j=1,\dots,m}$, $Y = \begin{pmatrix} u_{m_1} \\ \vdots \\ u_{m_m} \end{pmatrix}$

and $N(Y) = \begin{pmatrix} (f(u_m), e_1) \\ \vdots \\ (f(u_m), e_m) \end{pmatrix}$.

We can easily check that matrix M_ε is invertible. Indeed, setting $X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \neq 0$

and $X' = \displaystyle\sum_{i=1}^{m} x_i e_i$, we have

$$(M_\varepsilon X, X) = \sum_{j=1}^{m}\sum_{i=1}^{m} \left(A_\varepsilon^{-1} e_j, e_i\right) x_i x_j = \left(A_\varepsilon^{-1} X', X'\right) > 0,$$

due to the fact that A_ε^{-1} is a positive definite operator. Moreover, the matrix M is positive definite and $N(Y)$ depends continuously on Y. Applying Cauchy's theorem for a system of ordinary differential equations, it follows that there exists a time $t_m \in (0,T)$ and a unique solution Y for the equation $\dfrac{dY}{dt}+M_\varepsilon^{-1}MY+M_\varepsilon^{-1}N(Y)=0$ on the time interval $t \in [0,t_m[$. Based on the a priori estimates with respect to t that will be derived below for the solution $u_m(t)$, we deduce that any local solution of (2.14) is actually a global solution defined on the whole interval $[0,T]$.

Now, we give the a priori estimates for the solution $u_m(t)$.

A priori estimates: Multiplying (2.18) by u_m, we find

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{-1/2}u_m\|^2+\|\nabla u_m\|^2+(f(u_m),u_m)=0. \tag{2.20}$$

By (2.3), we have

$$(f(u_m),u_m) \geqslant c(\|u_m\|^2+\|u_m\|_{L^{2p}(\Omega)}^{2p})-c|\Omega|$$

and estimate (2.20) yields

$$\frac{d}{dt}\|A_\varepsilon^{-1/2}u_m\|^2+c(\|u_m\|_{H^1(\Omega)}^2+\|u_m\|_{L^{2p}(\Omega)}^{2p}) \leqslant c. \tag{2.21}$$

Multiplying (2.18) by $A_\varepsilon u_m$ and using (2.3), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_m\|^2+\|\Delta u_m\|^2+\varepsilon\|\nabla u_m\|^2 &\leqslant \kappa\|\nabla u_m\|^2+\kappa\varepsilon\|u_m\|^2\\ &\leqslant \kappa\left((-\Delta u_m,u_m)+\|u_m\|^2\right)\\ &\leqslant \kappa\|u_m\|^2+1/2\|\Delta u_m\|^2+\kappa^2/2\|u_m\|^2,\end{aligned}$$

which yields

$$\frac{d}{dt}\|u_m\|^2+\|\Delta u_m\|^2+2\varepsilon\|\nabla u_m\|^2 \leqslant c\|u_m\|^2. \tag{2.22}$$

Using the fact that $\|A_\varepsilon^{-1/2}u_m\| \leqslant c\|u_m\|_{H^1(\Omega)}$ and $\|u_m\| \leqslant c\|\Delta u_m\|$, estimates (2.21) and (2.22) give, for a positive constant α independent of ε,

$$\frac{d}{dt}\|A_\varepsilon^{-1/2}u_m\|^2+\alpha\|A_\varepsilon^{-1/2}u_m\|^2+c_\alpha\|u_m\|_{H^1(\Omega)}^2+c\|u_m\|_{L^{2p}(\Omega)}^{2p} \leqslant c, \tag{2.23}$$

and

$$\frac{d}{dt}\|u_m\|^2+\alpha\|u_m\|^2+c_\alpha\|\Delta u_m\|^2+2\varepsilon\|\nabla u_m\|^2 \leqslant c\|u_m\|^2, \tag{2.24}$$

where the constant $c_\alpha>0$ depends on α and is independent of ε.
Applying Gronwall's lemma, estimate (2.23) leads to

$$\|A_\varepsilon^{-1/2}u_m(t)\|^2+\int_0^t(\|u_m(s)\|_{H^1(\Omega)}^2+\|u_m(s)\|_{L^{2p}(\Omega)}^{2p})e^{-\alpha(t-s)}\mathrm{d}s \leqslant c\|A_\varepsilon^{-1/2}u_m(0)\|^2e^{-\alpha t}+c. \tag{2.25}$$

Integrating (2.21) with respect to $s \in [t, t+1]$ and using estimate (2.25), we obtain

$$\int_t^{t+1} (\|u_m(s)\|^2_{H^1(\Omega)} + \|u_m(s)\|^{2p}_{L^{2p}(\Omega)}) \mathrm{d}s \leqslant \|A_\varepsilon^{-1/2} u_m(t)\|^2 + c \leqslant c\|A_\varepsilon^{-1/2} u_m(0)\|^2 e^{-\alpha t} + c. \tag{2.26}$$

Estimates (2.25) and (2.26) give

$$\|A_\varepsilon^{-1/2} u_m(t)\|^2 + \int_t^{t+1} (\|u_m(s)\|^2_{H^1(\Omega)} + \|u_m(s)\|^{2p}_{L^{2p}(\Omega)}) \mathrm{d}s \leqslant c\|A_\varepsilon^{-1/2} u_m(0)\|^2 e^{-\alpha t} + c,$$

so that

$$u_m \in L^2(0,T;V) \cap L^{2p}(0,T;L^{2p}(\Omega)) \cap L^\infty(0,T;H^{-1}(\Omega)).$$

Applying Gronwall's Lemma and using estimate (2.25), relation (2.22) leads to

$$\begin{aligned} \|u_m(t)\|^2 &\leqslant \|u_m(0)\|^2 e^{-\alpha t} + c \int_0^t \|u_m(s)\|^2 e^{-\alpha(t-s)} \mathrm{d}s \\ &\leqslant c\|u_m(0)\|^2 e^{-\alpha t} + c, \end{aligned} \tag{2.27}$$

which implies that $u_m \in L^\infty(0,T;H)$. Now, integrating estimate (2.22) with respect to $s \in [t, t+1]$ and using (2.27), we obtain

$$\|u_m(t)\|^2 + \int_t^{t+1} (\|\Delta u_m(s)\|^2 + 2\varepsilon \|\nabla u_m(s)\|^2) \mathrm{d}s \leqslant c\|u_m(0)\|^2 e^{-\alpha t} + c, \tag{2.28}$$

where the positive constant c is independent of m. Integrating (2.22) with respect to $t \in [0,T]$ and using (2.27), we find

$$\int_0^T \|\Delta u_m(t)\|^2 dt \leqslant \|u_m(0)\|^2 + \int_0^T \|u_m(t)\|^2 dt \leqslant \|u_m(0)\|^2 + c\|u_m(0)\|^2 + cT$$

and we deduce that $u_m \in L^2(0,T;W)$.

We introduce the energy functional $\mathcal{E} : V \to \mathbb{R}$ defined by

$$\mathcal{E}(v) = \frac{1}{2}\|\nabla v\|^2 + \int_\Omega F(v) \mathrm{d}x.$$

Multiplying (2.18) by $\partial_t u_m$, we obtain

$$\|A_\varepsilon^{-1/2} \partial_t u_m\|^2 + \frac{d}{dt}\left(\frac{1}{2}\|\nabla u_m\|^2 + \int_\Omega F(u_m) \mathrm{d}x\right) = 0, \tag{2.29}$$

which shows that the energy $\mathcal{E}$ decays along the trajectories. Integrating equation (2.29) with respect to t, we get

$$\begin{aligned} \mathcal{E}(u_m(t)) + \int_0^t \|A_\varepsilon^{-1/2} \partial_t u_m(s)\|^2 \mathrm{d}s &\leqslant \frac{1}{2}\|\nabla u_m(0)\|^2 + \int_\Omega F(u_m(0)) \\ &\leqslant \|\nabla u_0\|^2 + \frac{3}{2} b_{2p} \int_\Omega u_m^{2p}(0) \mathrm{d}x + c \\ &\leqslant \|\nabla u_0\|^2 + c\|u_0\|^{2p}_{L^{2p}(\Omega)} + c. \end{aligned}$$

Thus, if $u_0 \in V \cap L^{2p}(\Omega)$, we have a solution u_m which verifies

$$u_m \in L^\infty(0,T;V \cap L^{2p}(\Omega)) \text{ and } \partial_t u_m \in L^2(0,T;H^{-1}(\Omega)),\ \forall T > 0.$$

Passage to the limit: We now pass to the limit as $m \to \infty$ and study the convergence of the sequence $(u_m)_m$. Let q satisfy $\dfrac{1}{2p} + \dfrac{1}{q} = 1$, which implies $q < 2$. According to the a priori estimates derived in the previous section, $\|u_m\|_{L^2(0,T;W)}$ is uniformly bounded and, consequently, $(u_m)_m$ is bounded in $L^q(0,T;L^q(\Omega))$. Moreover, thanks to (2.3) and the uniform boundedness of $(u_m)_m$ in $L^{2p}(0,T;L^{2p}(\Omega))$, $\|f(u_m)\|_{L^q(0,T;L^q(\Omega))}$ is uniformly bounded, which implies that $(A_\varepsilon^{-1}\partial_t u_m)_m$ is bounded in $L^q(0,T;L^q(\Omega))$. It follows that

$$\|\partial_t u_m\|_{L^q(0,T;W^{-2,q}(\Omega))} \leqslant c. \tag{2.30}$$

Starting from this point, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not relabeled. Thus, we observe that weak and weak star compactness results applied to the sequence $(u_m)_m$ entail that there exists a function u such that the following properties hold:

$$u_m \rightharpoonup u \text{ weakly in } L^q(0,T;W), \tag{2.31}$$

$$\partial_t u_m \rightharpoonup \partial_t u \text{ weakly in } L^q(0,T;W^{-2,q}(\Omega)), \tag{2.32}$$

as $m \to \infty$. It follows from (2.31), (2.32) and the Aubin-Lions compactness theorem that

$$u_m \to u \text{ strongly in } L^q(0,T;L^q(\Omega)). \tag{2.33}$$

Consequently $u_m(t,x) \to u(t,x)$ a.e. $(t,x) \in [0,T] \times \Omega$.
Moreover, we have

$$\left.\begin{array}{l} u_m(t,x) \to u(t,x) \text{ a.e.} \\ f \text{ is a continuous function} \end{array}\right\} \Longrightarrow \quad f(u_m(t,x)) \to f(u(t,x)) \text{ a.e.}$$

$$\left.\begin{array}{l} f(u_m(t,x)) \to f(u(t,x)) \text{ a.e.} \\ \|f(u_m)\|_{L^q(\Omega_T)} \leqslant \text{ constant} \end{array}\right\} \Longrightarrow \quad f(u_m) \rightharpoonup f(u) \text{ weakly in } L^q(\Omega_T),$$

where we have used the weak dominated convergence theorem. Finally, we deduce that $A_\varepsilon^{-1}\partial_t u_m \rightharpoonup A_\varepsilon^{-1}\partial_t u$ weakly in $L^q(\Omega_T)$. Thus, passing to the limit in (2.18), we obtain

$$A_\varepsilon^{-1}\partial_t u - \Delta u + f(u) = 0, \quad \text{ in } L^q(\Omega_T). \tag{2.34}$$

To prove that $u(0) = u_0$, we consider a test function $\psi \in C^1([0,T];L^{2p}(\Omega))$ such that
$\psi(T) = 0$. Multiplying (2.4) by ψ and integrating over $\Omega \times [0,T]$, we obtain

$$\int_0^T \left\langle A_\varepsilon^{-1}\partial_t u, \psi \right\rangle dt - \int_0^T \langle \Delta u, \psi \rangle \, dt + \int_0^T \langle f(u), \psi \rangle \, dt = 0. \tag{2.35}$$

Integrating by parts in (2.35), we have

$$-\int_0^T \left\langle A_\varepsilon^{-1}u, \partial_t\psi \right\rangle dt - \left\langle A_\varepsilon^{-1}u(0), \psi(0) \right\rangle - \int_0^T \langle \Delta u, \psi \rangle \, dt + \int_0^T \langle f(u), \psi \rangle \, dt = 0. \tag{2.36}$$

Multiplying (2.18) by ψ and integrating over $\Omega \times [0,T]$, we obtain

$$-\int_0^T \left\langle A_\varepsilon^{-1} u_m, \partial_t \psi \right\rangle dt - \left\langle A_\varepsilon^{-1} P_m u(0), \psi(0) \right\rangle - \int_0^T \langle \Delta u_m, \psi \rangle \, dt + \int_0^T \langle f(u_m), \psi \rangle \, dt = 0. \tag{2.37}$$

Having $\psi \in L^{2p}(0,T; L^{2p}(\Omega))$ and $\dfrac{\partial \psi}{\partial t} \in L^2(0,T;H)$ we deduce that

$$-\int_0^T \langle \Delta u_m, \psi \rangle \, dt \to -\int_0^T \langle \Delta u, \psi \rangle \, dt$$

and

$$\int_0^T \left\langle A_\varepsilon^{-1} u_m, \partial_t \psi \right\rangle dt \to \int_0^T \left\langle A_\varepsilon^{-1} u, \partial_t \psi \right\rangle dt.$$

Then, passing to the limit in (2.37) as $m \to \infty$, we obtain

$$-\int_0^T \left\langle A_\varepsilon^{-1} u, \partial_t \psi \right\rangle dt - \left\langle A_\varepsilon^{-1} u_0, \psi(0) \right\rangle - \int_0^T \langle \Delta u, \psi \rangle \, dt + \int_0^T \langle f(u), \psi \rangle \, dt = 0. \tag{2.38}$$

We deduce from (2.36) and (2.38) that $A_\varepsilon^{-1} u(0) = A_\varepsilon^{-1} u_0$, which implies $u(0) = u_0$.

2.4 Additional regularity

In this section, we will derive some additional regularity for the solution $u(t)$. We have the following theorem.

Theorem 2.4.1. *Let us take $u_0 \in W$. Then, there exists a unique solution u of problem (2.4) with initial datum u_0 such that*

$$u \in L^\infty([0,T]; H^2(\Omega)) \cap L^2([0,T]; H^4(\Omega)). \tag{2.39}$$

Furthermore, we have

$$\|u\|_{H^2(\Omega)} \leqslant Q(\|u_0\|_{H^2(\Omega)}) e^{-\alpha t} + c, \tag{2.40}$$

where the constants $c, \alpha > 0$ and the monotonic function Q are independent of ε.

The proof is based on a priori estimates that we formally derive below, a rigorous justification being based on the classical Galerkin method.

Lemma 2.4.2. *Let $u(t)$ be a solution of (2.4) with $u_0 \in W$. Then, there exists a time $T_0 = T_0(\|u_0\|_{H^2(\Omega)})$, $0 < T_0 < 1/2$, and a monotonic function Q such that*

$$\|\Delta u(t)\| \leqslant Q(\|u_0\|_{H^2(\Omega)}), \quad t \leqslant T_0(\|u_0\|_{H^2(\Omega)}). \tag{2.41}$$

Proof:
Multiplying equation (2.4) by $\Delta^2 A_\varepsilon u(t)$, we obtain the following inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \|\Delta^2 u\|^2 + \varepsilon \|\nabla \Delta u\|^2 &= -\left(A_\varepsilon f(u), \Delta^2 u\right) \\ &\leqslant \frac{1}{2}\|A_\varepsilon f(u)\|^2 + \frac{1}{2}\|\Delta^2 u\|^2 \\ &\leqslant \frac{1}{2} c \|f(u)\|^2_{H^2(\Omega)} + \frac{1}{2}\|\Delta^2 u\|^2, \end{aligned} \tag{2.42}$$

where c is independent of ε. Taking into account that $f \in C^2(\Omega)$ and that $H^2(\Omega) \subset C(\overline{\Omega})$, we can prove that there exists a monotonic functions Q (depending on f) such that

$$\|f(u)\|^2_{H^2(\Omega)} \leqslant Q\left(\|u\|^2_{H^2(\Omega)}\right) \leqslant Q\left(\|\Delta u\|^2\right). \tag{2.43}$$

Indeed, we have for $n \leqslant 3$ (where n is the space dimension)

$$\|f'(u)\|^2_{L^\infty(\Omega)} \leqslant Q_{f'}(\|u\|^2_{L^\infty(\Omega)}) \leqslant Q_{f'}(\|u\|^2_{H^2(\Omega)}), \text{ since } f' \in C(\overline{\Omega}),$$

$$\|f''(u)\|^2_{L^\infty(\Omega)} \leqslant Q_{f''}(\|u\|^2_{L^\infty(\Omega)}) \leqslant Q_{f''}(\|u\|^2_{H^2(\Omega)}), \text{ since } f'' \in C(\overline{\Omega}),$$

and

$$\begin{aligned}
\|f(u)\|^2_{H^2(\Omega)} \leqslant & c\|\Delta f(u)\|^2 \\
\leqslant & c(\|f'(u)\Delta u\|^2 + \|f''(u)\nabla u \cdot \nabla u\|^2) \\
\leqslant & c(\|f'(u)\|^2_{L^\infty(\Omega)}\|\Delta u\|^2 + \|f''(u)\|^2_{L^\infty(\Omega)}\|\nabla u\|^4_{L^4(\Omega)}) \\
\leqslant & c(\|f'(u)\|^2_{L^\infty(\Omega)}\|\Delta u\|^2 + \|f''(u)\|^2_{L^\infty(\Omega)}\|\Delta u\|^4) \\
\leqslant & Q(\|u\|^2_{H^2(\Omega)}),
\end{aligned} \tag{2.44}$$

for some monotonic increasing functions $Q_{f'}$ and $Q_{f''}$.
Thus, replacing inequality (2.43) in (2.42), the function $y(t) := \|\Delta u(t)\|^2$ satisfies the inequality

$$y'(t) \leqslant Q\left(y(t)\right).$$

Let $z(t)$ be a solution of the following equation:

$$z'(t) = Q\left(z(t)\right),\ z(0) = y(0) = \|\Delta u(0)\|^2.$$

Due to the comparison principle, there exists a time $T_0(\|u_0\|_{H^2(\Omega)}) \in (0, 1/2)$ such that we have

$$y(t) \leqslant z(t),\ \ \forall t \leqslant T_0(\|u_0\|_{H^2(\Omega)}). \tag{2.45}$$

Then, the lemma is an immediate consequence of (2.45). □

Lemma 2.4.3. *Let the above assumptions hold and let T_0 be the same as in Lemma 2.4.2. Then, the following estimate holds:*

$$t\|A_\varepsilon^{-1/2}\partial_t u(t)\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}),\ t \in (0, T_0], \tag{2.46}$$

for some monotonic function Q.

Proof:
Multiplying (2.4) by $\partial_t u(t)$ and integrating over Ω, we obtain

$$\|A_\varepsilon^{-1/2}\partial_t u\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 \leqslant |(f(u), \partial_t u)| \leqslant c\|f(u)\|^2_{H^1(\Omega)} + 1/2\|A_\varepsilon^{-1/2}\partial_t u\|^2. \tag{2.47}$$

Integrating (2.47) over $[0, T_0]$ and taking into account (2.41) and the fact that $H^2(\Omega) \subset C(\overline{\Omega})$, we have

$$\begin{aligned}
\int_0^{T_0} \|A_\varepsilon^{-1/2}\partial_t u(t)\|^2 dt \leqslant & c\int_0^{T_0} \|f(u(t))\|^2_{H^1(\Omega)} dt + \|\nabla u(0)\|^2 \\
\leqslant & Q(\|u_0\|_{H^2(\Omega)}) + \|\nabla u(0)\|^2 \\
\leqslant & Q(\|u_0\|_{H^2(\Omega)}),
\end{aligned} \tag{2.48}$$

where we have used the fact that

$$\|f(u(t))\|^2_{H^1(\Omega)} \leqslant Q(\|u(t)\|_{H^2(\Omega)}) \leqslant Q(\|u_0\|_{H^2(\Omega)}),\ \forall t \in [0, T_0],$$

for some monotonic function Q. Differentiating (2.4) with respect to t and setting $\theta(t) = \partial_t u(t)$, we find

$$\begin{cases} A_\varepsilon^{-1}\partial_t\theta - \Delta\theta + f'(u(t))\theta = 0, \\ \theta|_{\partial\Omega} = 0. \end{cases} \tag{2.49}$$

Multiplying (2.49) by $t\theta(t)$, integrating over Ω and using the fact that $f'(u) \geqslant -\kappa$, we obtain

$$\begin{aligned} \frac{d}{dt}(t\|A_\varepsilon^{-1/2}\theta\|^2) + 2t\|\nabla\theta\|^2 \leqslant& 2\kappa t\|\theta\|^2 + \|A_\varepsilon^{-1/2}\theta\|^2 \\ \leqslant& ct\|A_\varepsilon^{-1/2}\theta\|\|\nabla\theta\| + \|A_\varepsilon^{-1/2}\theta\|^2 \\ \leqslant& t\|\nabla\theta\|^2 + c(t+1)\|A_\varepsilon^{-1/2}\theta\|^2. \end{aligned}$$

Hence, we have

$$\frac{d}{dt}(t\|A_\varepsilon^{-1/2}\theta\|^2) \leqslant c(t+1)\|A_\varepsilon^{-1/2}\theta\|^2. \tag{2.50}$$

Applying Gronwall's lemma to estimate (2.50) over $(0, t)$ for $t \leqslant T_0 \leqslant 1$ and using (2.48), we deduce that

$$t\|A_\varepsilon^{-1/2}\theta(t)\|^2 \leqslant c\int_0^t \|A_\varepsilon^{-1/2}\theta(s)\|^2 \mathrm{d}s \leqslant Q(\|u_0\|_{H^2(\Omega)}) \tag{2.51}$$

and Lemma 2.4.3 is proven. □

Lemma 2.4.4. *Let $u(t)$ be a solution of equation (2.4) and let $t \geqslant T_0$, where T_0 is the same as in Lemma 2.4.2. Then, the following estimate holds uniformly with respect to ε:*

$$\|A_\varepsilon^{-1/2}\partial_t u(t)\|^2 + \|u(t)\|^2_{H^2(\Omega)} + \int_t^{t+1} \|\partial_t u(s)\|^2_{H^1(\Omega)} \mathrm{d}s \leqslant e^{K_1 t} Q(\|u_0\|_{H^2(\Omega)}),\ \forall t \geqslant T_0,$$

where K_1 is a positive constant and Q is some monotonic function.

Proof:
Multiplying (2.49) by $\theta(t)$, integrating over Ω and using the fact that $f'(u) \geqslant -\kappa$, we have

$$\begin{aligned} \frac{d}{dt}\|A_\varepsilon^{-1/2}\theta\|^2 + 2\|\nabla\theta\|^2 \leqslant& 2\kappa\|\theta\|^2 \\ \leqslant& 2c\kappa\|A_\varepsilon^{-1/2}\theta\|\|\nabla\theta\| \\ \leqslant& \|\nabla\theta\|^2 + c^2\kappa^2\|A_\varepsilon^{-1/2}\theta\|^2. \end{aligned} \tag{2.52}$$

Applying Gronwall's lemma to estimate (2.52) over (T_0, t), we obtain

$$\|A_\varepsilon^{-1/2}\theta(t)\|^2 + \int_t^{t+1} \|\nabla\theta(s)\|^2 \mathrm{d}s \leqslant e^{K_1 t}\|A_\varepsilon^{-1/2}\theta(T_0)\|^2,\ t \geqslant T_0, \tag{2.53}$$

for some positive constant K_1. Using Lemma 2.4.3, estimate (2.53) gives

$$\|A_\varepsilon^{-1/2}\partial_t u(t)\|^2 + \int_t^{t+1} \|\partial_t u(s)\|^2_{H^1(\Omega)} \mathrm{d}s \leqslant e^{K_1 t} Q(\|u_0\|_{H^2(\Omega)}),\ t \geqslant T_0. \tag{2.54}$$

Interpreting the parabolic equation (2.4) as an elliptic boundary value problem,

$$\Delta u(t) - f(u(t)) = h(t) := A_\varepsilon^{-1}\partial_t u(t),\ u(t)|_{\partial\Omega} = 0, \tag{2.55}$$

for every fixed $t \geqslant T_0$, estimate (2.54) implies that

$$\|h(t)\|^2 = \|A_\varepsilon^{-1/2}\partial_t u(t)\|^2 \leqslant e^{K_1 t} Q(\|u_0\|_{H^2(\Omega)}). \tag{2.56}$$

Multiplying (2.55) by $u(t)$, using the fact that $f(u)u \geqslant -c$ and integrating over Ω, we obtain

$$\|\nabla u(t)\|^2 \leqslant c(1 + \|h(t)\|^2). \tag{2.57}$$

Multiplying (2.55) by $\Delta u(t)$, using the fact that $f'(u) \geqslant -\kappa$ and integrating over Ω, we find

$$\|\Delta u(t)\|^2 \leqslant c(\|\nabla u(t)\|^2 + \|h(t)\|^2)). \tag{2.58}$$

We deduce from estimates (2.56), (2.57) and (2.58) that

$$\|u(t)\|^2_{H^2(\Omega)} \leqslant c\|\Delta u(t)\|^2 \leqslant ce^{K_1 t} Q(\|u_0\|_{H^2(\Omega)}) \tag{2.59}$$

and the lemma is proven. □

Corollary 2.4.5. *Let the above assumptions hold and let $u(t)$ be a solution of problem (2.4). Then, the following estimate holds:*

$$\|u(t)\|_{H^2(\Omega)} \leqslant Q(\|u(0)\|_{H^2(\Omega)})e^{Kt}, \tag{2.60}$$

for all $t \geqslant 0$ and where the positive constant K and the monotonic function Q are independent of ε.

Proof:
We have proved that estimate (2.60) is true for $t \leqslant T_0(\|u_0\|_{H^2(\Omega)})$ in Lemma 2.4.2 and for $t \geqslant T_0(\|u_0\|_{H^2(\Omega)})$ in Lemma 2.4.4, which completes the proof. □

Lemma 2.4.6. *Let $u(t)$ be a solution of problem (2.4). Then, the following estimate is valid, uniformly with respect to ε:*

$$\|u(1)\|_{H^2(\Omega)} \leqslant Q(\|u_0\|^2), \tag{2.61}$$

for some monotonic function Q that is independent of ε.

Proof:
Estimate (2.28) yields

$$\int_0^1 \|u(s)\|_{H^2(\Omega)} \mathrm{d}s \leqslant c\|u(0)\|^2 + c.$$

It follows that there exists a time $T \in [0,1]$ such that

$$\|u(T)\|_{H^2(\Omega)} \leqslant c\|u(0)\|^2 + c. \tag{2.62}$$

Applying estimate (2.60) starting with the time $t = T$ instead of $t = 0$ and using (2.62), we obtain the desired inequality. □

To complete the proof of Theorem 2.4.1, we proceed as follows. We have from estimate (2.28) that for every $t \geqslant 1$, there exists a time $t_* \in [t-1, t]$ such that:

$$\|\Delta u(t_*)\|^2 \leqslant c\|u(0)\|^2 e^{-\alpha t} + c.$$

Consequently, for $t' \in [0,1]$ such that $t = t_* + t'$, estimate (2.60) yields:

$$\begin{aligned}\|\Delta u(t)\|^2 = \|\Delta u(t_* + t')\|^2 &\leqslant Q(\|\Delta u(t_*)\|^2)e^{Kt'} \\ &\leqslant cQ(\|\Delta u(t_*)\|^2)e^{-\alpha t'} \\ &\leqslant cQ(c\|u(0)\|^2 e^{-\alpha t} + c) \\ &\leqslant Q(\|u(0)\|^2_{H^2(\Omega)})e^{-\alpha t} + c,\end{aligned}$$

for some monotonic function Q. Theorem 2.4.1 is thus proven.

Remark 2.4.7. Theorem 2.4.1 is also true for every function f such that $f \in C^2(\mathbb{R})$, $f(0) = 0$, $f'(s) \geqslant -\kappa$ and $f(s)s \geqslant -c$, $s \in \mathbb{R}$ and κ, $c \geqslant 0$.

The next two Lemmata give additional regularity for the time derivative of u.

Lemma 2.4.8. *Let the assumptions of Theorem 2.4.1 hold. Then, the solution $u(t)$ of problem (2.4) satisfies*

$$\|\partial_t u(t)\|^2 + \int_t^{t+1} \|\partial_t u(s)\|^2_{H^2(\Omega)} \mathrm{d}s \leqslant Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha t} + c,\ t \geqslant 1, \tag{2.63}$$

where the positive constants c, α and the function Q are independent of ε.

Proof:
Multiplying (2.49) by $(t - T_0)A_\varepsilon\theta(t)$, where T_0 is the same as in Lemma 2.4.3 and $\theta = \partial_t u$, and integrating over Ω, we find

$$\begin{aligned}\frac{d}{dt}\left((t-T_0)\frac{\|\theta(t)\|^2}{2}\right) + \alpha'(t-T_0)\frac{\|\theta(t)\|^2}{2} + c_{\alpha'}(t-T_0)\|\Delta\theta(t)\|^2 + \varepsilon(t-T_0)\|\nabla\theta(t)\|^2 \\ \leqslant (1 + \kappa\varepsilon(t-T_0))\|\theta(t)\|^2 + c(t-T_0)\|f'(u(t))\theta(t)\|^2 \\ \leqslant c(t-T_0)(\|\theta(t)\|^2 + \|f'(u(t))\theta(t)\|^2) := c(t-T_0)h_u(t),\end{aligned} \tag{2.64}$$

for appropriate positive constants α' and c'_α. Applying Gronwall's lemma to (2.64) over (T_0, t), we obtain

$$\begin{aligned}(t-T_0)\|\theta(t)\|^2 e^{\alpha' t} &\leqslant c\int_{T_0}^t (s-T_0)h_u(s)e^{\alpha' s}\mathrm{d}s \\ &\leqslant c(t-T_0)\int_{T_0}^t h_u(s)e^{\alpha' s}\mathrm{d}s,\end{aligned}$$

so that

$$\|\theta(t)\|^2 \leqslant c\int_{T_0}^{t} h_u(s)e^{-\alpha' t}e^{\alpha' s}\mathrm{d}s = c\int_{T_0}^{t} h_u(s)e^{-\alpha'(t-s)}\mathrm{d}s. \tag{2.65}$$

To estimate $\displaystyle\int_{T_0}^{t} h_u(s)e^{-\alpha'(t-s)}\mathrm{d}s$, we proceed as follows.
Using Theorem 2.4.1 and estimate (2.54), we find

$$\begin{aligned}\int_t^{t+1} \|\partial_t u(s)\|^2_{H^1(\Omega)}\mathrm{d}s &\leqslant Q(\|u(t-T_0)\|_{H^2(\Omega)})e^{K_1T_0}\\ &\leqslant e^{K_1T_0}(Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha(t-T_0)}+c)\\ &\leqslant Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha t}+c,\end{aligned} \tag{2.66}$$

and using the fact that $W \subset C(\overline{\Omega})$, we obtain

$$\int_t^{t+1} h_u(s)\mathrm{d}s \leqslant Q(\sup_{s\in[t,t+1]}\|u(s)\|_{L^\infty(\Omega)})\int_t^{t+1}\|\partial_t u(s)\|^2_{H^1(\Omega)}\mathrm{d}s \leqslant Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha t}+c, \tag{2.67}$$

for $t \geqslant T_0$ and for an appropriate function Q and constants $c, \alpha > 0$ which are independent of ε.
We have

$$\begin{aligned}&\int_{T_0}^{t} h_u(s)e^{-\alpha'(t-s)}\mathrm{d}s = e^{-\alpha' t}\int_{T_0}^{t} h_u(s)e^{\alpha' s}\mathrm{d}s\\
\leqslant& e^{-\alpha' t}\left(\int_{T_0}^{1} h_u(s)e^{\alpha' s}\mathrm{d}s + \int_1^2 h_u(s)e^{\alpha' s}\mathrm{d}s + \cdots + \int_{[t]-1}^{[t]} h_u(s)e^{\alpha' s}\mathrm{d}s + \int_{[t]}^{[t]+1} h_u(s)e^{\alpha' s}\mathrm{d}s\right)\\
\leqslant& e^{-\alpha' t}\left(e^{\alpha'}\int_{T_0}^{1} h_u(s)\mathrm{d}s + e^{2\alpha'}\int_1^2 h_u(s)\mathrm{d}s + \cdots + e^{\alpha'[t]}\int_{[t]-1}^{[t]} h_u(s)\mathrm{d}s + e^{\alpha'([t]+1)}\int_{[t]}^{[t]+1} h_u(s)\mathrm{d}s\right)\\
\leqslant& e^{-\alpha' t}(e^{\alpha'}(Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha T_0}+c) + e^{2\alpha'}(Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha}+c) + \cdots\\
&+ e^{\alpha'[t]}(Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha([t]-1)}+c) + e^{\alpha'([t]+1)}(Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha[t]}+c))\\
\leqslant& e^{-\alpha' t}(e^{\alpha'}(Q(\|u_0\|_{H^2(\Omega)})+c) + e^{2\alpha'}(Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha}+c) + \cdots\\
&+ e^{\alpha'[t]}(Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha([t]-1)}+c) + e^{\alpha'([t]+1)}(Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha[t]}+c))\\
\leqslant& e^{-\alpha' t}e^{\alpha'}Q(\|u_0\|_{H^2(\Omega)})\left(\frac{e^{(\alpha'-\alpha)([t]+1)}-1}{e^{(\alpha'-\alpha)}-1}\right) + ce^{-\alpha' t}e^{\alpha'}\left(\frac{e^{\alpha'([t]+1)}-1}{e^{\alpha'}-1}\right)\\
\leqslant& Q(\|u_0\|_{H^2(\Omega)})e^{-\alpha t}+c,\end{aligned} \tag{2.68}$$

where $\alpha \leqslant \alpha'$.
From estimates (2.65) and (2.68), we deduce the lemma. □

Lemma 2.4.9. *Let the assumptions of Theorem 2.4.1 hold. Then, we have the following:*

$$\|\partial_t u(t)\|^2_{H^1(\Omega)} + \int_t^{t+1}\|\partial_t^2 u(s)\|^2_{H^{-1}(\Omega)}\mathrm{d}s \leqslant Q(\|u(0)\|_{H^2(\Omega)})e^{-\alpha t}+c,$$

where $t \geqslant 2$ and the constants $c, \alpha > 0$ and the function Q are independent of ε.

Proof:
Multiplying (2.49) by $(t-T_*)\partial_t\theta(t)$, where $\theta=\partial_t u$ and $t\geqslant T_*\geqslant 1$ is arbitrary, using (2.8) and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[(t-T_*)\|\nabla\theta(t)\|^2]+(t-T_*)\|\partial_t\theta(t)\|^2_{H^{-1}(\Omega)}\\ =&\frac{1}{2}\|\nabla\theta(t)\|^2-(t-T_*)(f'(u(t))\theta(t),\partial_t\theta(t)):=H_u(t).\end{aligned}\tag{2.69}$$

We have

$$\begin{aligned}&(f'(u(t))\theta(t),\partial_t\theta(t))\leqslant\|f'(u(t))\theta(t)\|_{H^1(\Omega)}\|\partial_t\theta(t)\|_{H^{-1}(\Omega)}\\ &\leqslant(\|f'(u(t))\|_{L^\infty(\Omega)}\|\nabla\theta(t)\|+\|\nabla f'(u(t))\|_{L^4(\Omega)}\|\theta(t)\|_{L^4(\Omega)})\|\partial_t\theta(t)\|_{H^{-1}(\Omega)}\\ &\leqslant(\|f'(u(t))\|_{L^\infty(\Omega)}+\|\nabla f'(u(t))\|_{L^4(\Omega)})\|\theta(t)\|_{H^1(\Omega)}\|\partial_t\theta(t)\|_{H^{-1}(\Omega)}\\ &\leqslant(\|f'(u(t))\|_{L^\infty(\Omega)}+\|f''(u(t))\|_{L^\infty(\Omega)}\|u(t)\|_{H^2(\Omega)})\|\theta(t)\|_{H^1(\Omega)}\|\partial_t\theta(t)\|_{H^{-1}(\Omega)}.\end{aligned}\tag{2.70}$$

It follows from Theorem 2.4.1 and estimates (2.63), (2.70) and from the fact that $W\subset C(\overline{\Omega})$ and $H^1(\Omega)\subset L^6(\Omega)\subset L^4(\Omega)$ $(n\leqslant 3)$ that

$$\int_{T_*}^{T_*+s}H_u(t)dt\leqslant(1+s)(Q(\|u(0)\|_{H^2(\Omega)})e^{-\alpha T_*}+c)+\frac{1}{2}\int_{T_*}^{T_*+s}(t-T_*)\|\partial_t\theta(t)\|^2_{H^{-1}(\Omega)}dt,\tag{2.71}$$

where $s\in[0,2]$ and Q is an appropriate function independent of ε.
Integrating (2.69) with respect to $t\in[T_*,T_*+s]$ and using (2.71), we find

$$s\|\theta(T_*+s)\|^2_{H^1(\Omega)}+\int_{T_*}^{T_*+s}(t-T_*)\|\partial_t\theta(t)\|^2_{H^{-1}(\Omega)}dt\leqslant Q(\|u(0)\|_{H^2(\Omega)})e^{-\alpha T_*}+c,$$

for $T_*\geqslant 1$ and $s\in[0,2]$. Thus, for $T_*:=t-1\geqslant 1$ and $s=1$, we deduce that

$$\begin{aligned}\|\theta(t)\|^2_{H^1(\Omega)}+\int_{t-1}^{t}\|\partial_t\theta(s)\|^2_{H^{-1}(\Omega)}\mathrm{d}s&\leqslant Q(\|u(0)\|_{H^2(\Omega)})e^{-\alpha(t-1)}+c\\ &\leqslant cQ(\|u(0)\|_{H^2(\Omega)})e^{-\alpha t}+c.\end{aligned}\tag{2.72}$$

We thus deduce the lemma from estimates (2.72). □

2.5 Existence of the global attractor

In this section, we will study the long time behavior of the solutions of (2.4). We have the following.

Lemma 2.5.1. *Problem* (2.4) *generates the following semigroup on the phase space* H*:*

$$\begin{aligned}S_\varepsilon(t):H&\longrightarrow H\\ u_0&\longmapsto S_\varepsilon(t)u_0=u(t),\ t\geqslant 0,\end{aligned}$$

where $u(t)$ is the unique solution of problem (2.4), at time t, with initial datum u_0. Furthermore, this semigroup is Lipschitz continuous in the $H^{-1}(\Omega)-$topology,

$$\|A_\varepsilon^{-1/2}(S_\varepsilon(t)u_1-S_\varepsilon(t)u_2)\|^2+\int_t^{t+1}\|S_\varepsilon(s)u_1-S_\varepsilon(s)u_2\|^2_{H^1(\Omega)}\mathrm{d}s\leqslant ce^{Ct}\|A_\varepsilon^{-1/2}(u_1-u_2)\|^2, \tag{2.73}$$

for any $u_1,\ u_2\in H$, where c and C are positive constants independent of t and ε. Thus, $S_\varepsilon(t)$ can be uniquely extended, by continuity, to a semigroup, still denoted by $S_\varepsilon(t)$, acting on $H^{-1}(\Omega)$.

To prove the existence of the global attractor, we apply the following theorem.

Theorem 2.5.2. *(See [50]) If $S_\varepsilon(t)$ is dissipative in $H^{-1}(\Omega)$ and B is a compact absorbing set in $H^{-1}(\Omega)$, then $S_\varepsilon(t)$ has the global attractor $\mathcal{A}_\varepsilon=\omega(B)$.*

We have the following theorem.

Theorem 2.5.3. *The semigroup $S_\varepsilon(t)$ possesses the global attractor $\mathcal{A}_\varepsilon$ in $H^{-1}(\Omega)$ which is bounded in W.*

We devide the proof of this theorem into several propositions.

Proposition 2.5.4. *Problem (2.4) has an absorbing set in $H^{-1}(\Omega)$. More precisely, there exists a positive constant ρ_0 and a time $t_0:=t_0(\|u_0\|_{-1})$ such that*

$$\|A_\varepsilon^{-1/2}u(t)\|\leqslant\rho_0,\ \textit{for all } t\geqslant t_0.$$

Proof:
We multiply (2.4) by u, integrate over Ω and obtain

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{-1/2}u\|^2+\|\nabla u\|^2+(f(u),u)=0. \tag{2.74}$$

Using (2.3) with $\alpha=p-\dfrac{3}{2}$, we have

$$\begin{aligned}0\leqslant\int_\Omega|f(u)|\mathrm{d}x&\leqslant pb_{2p}\int_\Omega u^{2p}\mathrm{d}x-\frac{3}{2}b_{2p}\int_\Omega u^{2p}\mathrm{d}x+c|\Omega|\\&\leqslant(f(u),u)-\frac{3}{2}b_{2p}\int_\Omega u^{2p}\mathrm{d}x+c.\end{aligned}$$

Thus, using (2.3), we get

$$\begin{aligned}\|\nabla u\|^2+(f(u),u)&\geqslant\|\nabla u\|^2+\frac{3}{2}b_{2p}\int_\Omega u^{2p}\mathrm{d}x-c\\&\geqslant\frac{1}{2}\|\nabla u\|^2+\int_\Omega F(u)\mathrm{d}x-c\\&=\mathcal{E}(u)-c.\end{aligned}$$

Therefore, (2.74) yields

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{-1/2}u\|^2+\mathcal{E}(u)\leqslant c. \tag{2.75}$$

Using (2.3) and the fact that

$$\|A_\varepsilon^{-1/2}u\|^2 \leqslant c\|u\|^2 \leqslant c\|\nabla u\|^2,\ \forall u \in V,$$

we have the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{-1/2}u\|^2 + \frac{1}{2c}\|A_\varepsilon^{-1/2}u\|^2 + \frac{1}{2}b_{2p}\int_\Omega u^{2p}\mathrm{d}x \leqslant c.$$

By Gronwall's lemma, we find

$$\begin{aligned}\|A_\varepsilon^{-1/2}u(t)\|^2 \leqslant& \|A_\varepsilon^{-1/2}u(0)\|^2 e^{-t/c} + c(1-e^{-t/c})\\ \leqslant& c\|(-\Delta)^{-1/2}u(0)\|^2 e^{-t/c} + c(1-e^{-t/c}).\end{aligned}$$

It follows that if $t \geqslant t_0(\|(-\Delta)^{-1/2}u(0)\|^2) \equiv c\ln\left(c\|(-\Delta)^{-1/2}u(0)\|^2\right)$, then

$$\|A_\varepsilon^{-1/2}u(t)\|^2 \leqslant \rho_0^2.$$

□

Proposition 2.5.5. *For an arbitrary constant $r > 0$, there exists a positive constant ρ_V depending on r and a time $t_1 := t_1(\|u_0\|_{-1}, r)$ such that*

$$\|\nabla u(t)\| \leqslant \rho_V(r) \text{ and } \|u(t)\|^p_{L^{2p}(\Omega)} \leqslant \frac{\rho_V(r)}{\sqrt{b_{2p}}} \tag{2.76}$$

for all $t \geqslant t_0 + r := t_1$.

Proof:
Fixing a constant $r > 0$ and integrating (2.75) with respect to t, we obtain

$$\int_t^{t+r} \mathcal{E}(u(s))\mathrm{d}s \leqslant \frac{1}{2}\rho_0^2 + rc, \quad \forall t \geqslant t_0, \tag{2.77}$$

where t_0 is as in Proposition 2.5.4. Since $\mathcal{E}$ decays along the trajectories, we conclude from (2.77) that

$$\mathcal{E}(u(t+r)) \leqslant \frac{1}{2r}\rho_0^2 + c, \quad \forall t \geqslant t_0.$$

By the definition of $\mathcal{E}$ and (2.3), we have

$$\|\nabla u(t)\|^2 + b_{2p}\int_\Omega u^{2p}(x,t)\mathrm{d}x \leqslant 2c|\Omega| + \frac{1}{r}\rho_0^2 + 2c := \rho_V^2, \tag{2.78}$$

for all $t \geqslant t_0 + r := t_1$. The proposition is thus proven. □

Proposition 2.5.6. *For an arbitrary constant $r > 0$, there exists a positive constant ρ depending on r and a time $t_2 := t_2(\|u_0\|_{-1}, r)$ such that*

$$\|\Delta u(t)\| \leqslant \rho(r) \text{ for all } t \geqslant t_1 + r := t_2.$$

Proof:
Multiplying (2.4) by $\partial_t u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{d}{dt}\int_\Omega F(u)\mathrm{d}x + \|A_\varepsilon^{-1/2}\partial_t u\|^2 = 0. \tag{2.79}$$

Using (2.76), we find

$$\begin{aligned}
&\int_t^{t+r} \|A_\varepsilon^{-1/2}\partial_t u(s)\|^2\mathrm{d}s + \frac{1}{2}\|\nabla u(t+r)\|^2 + \frac{b_{2p}}{2}\int_\Omega u^{2p}(t+r)\mathrm{d}x \\
&\leqslant \frac{1}{2}\|\nabla u(t)\|^2 + \frac{b_{2p}}{2}\int_\Omega u^{2p}(t)\mathrm{d}x + c \\
&\leqslant \frac{3}{2}\rho_V^2 + c, \quad \forall t \geqslant t_1.
\end{aligned} \tag{2.80}$$

Multiplying (2.49) by $\partial_t u$ and using the fact that $f'(u) \geqslant -\kappa$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{-1/2}\partial_t u\|^2 + \|\nabla\partial_t u\|^2 \leqslant \kappa\|\partial_t u\|^2 \leqslant c\|A_\varepsilon^{-1/2}\partial_t u\|\|\nabla\partial_t u\|,$$

which yields

$$\frac{d}{dt}\|A_\varepsilon^{-1/2}\partial_t u\|^2 \leqslant c\|A_\varepsilon^{-1/2}\partial_t u\|^2, \tag{2.81}$$

for some positive constant c. Using (2.80), (2.81) and the uniform Gronwall lemma, we deduce that

$$\|A_\varepsilon^{-1/2}\partial_t u(t)\|^2 \leqslant c(r), \quad \forall t \geqslant t_1 + r. \tag{2.82}$$

We rewrite (2.4) in the following form:

$$-\Delta u + f(u) = -A_\varepsilon^{-1}\partial_t u. \tag{2.83}$$

Multiplying (2.83) by $-\Delta u(t)$ and using the fact that $f'(u) \geqslant -\kappa$, we obtain

$$\begin{aligned}
\|\Delta u(t)\|^2 \leqslant& \kappa\|\nabla u(t)\|^2 + (A_\varepsilon^{-1}\partial_t u(t), \Delta u(t)) \\
\leqslant& \kappa\|\nabla u(t)\|^2 + \|A_\varepsilon^{-1}\partial_t u(t)\|\|\Delta u(t)\| \\
\leqslant& \kappa\|\nabla u(t)\|^2 + c\|A_\varepsilon^{-1/2}\partial_t u(t)\|\|\Delta u(t)\|,
\end{aligned}$$

which yields

$$\|\Delta u(t)\|^2 \leqslant 2\kappa\|\nabla u(t)\|^2 + c\|A_\varepsilon^{-1/2}\partial_t u(t)\|^2. \tag{2.84}$$

Using (2.78) and (2.82), estimate (2.84) gives

$$\|\Delta u(t)\| \leqslant \rho(r), \quad \forall t \geqslant t_2 := t_1 + r,$$

for some positive constant $\rho(r)$ which depend on r and Proposition 2.5.6 is proven. □

Let

$$B := \left\{u \in W, \|\Delta u\|^2 \leqslant \rho\right\}, \tag{2.85}$$

where ρ is the same as in Proposition 2.5.6. Then, B is a compact absorbing set in $H^{-1}(\Omega)$ since it is an absorbing set in W and W is compactly embedded into $H^{-1}(\Omega)$. Applying Theorem 2.5.2, we deduce the existence of the global attractor $\mathcal{A}_\varepsilon = \omega(B)$ and Theorem 2.5.3 is thus proven.

2.6 Existence of an exponential attractor

To prove the existence of an exponential attractor, we apply the following theorem.

Theorem 2.6.1. *(See [44]) Let E and E_1 be two Hilbert spaces such that E_1 is compactly embedded into E and $S(t) : X \to X$ be a semigroup acting on a closed subset X of E. We assume that*

1. $\forall x_1,\ x_2 \in X,\ \forall t > 0,$

$$\|S(t)x_1 - S(t)x_2\|_{E_1} \leqslant h(t)\|x_1 - x_2\|_E,$$

where the function h is continuous;

2. $(t, x) \mapsto S(t)x$ *is uniformly Hölder continuous in the topology of E on $[0, T] \times B$,*
$\forall T > 0,\ \forall B \subset X$ *bounded.*

Then, $S(t)$ possesses an exponential attractor on X.

In order to apply this result to the semigroup $S_\varepsilon(t)$ associated with problem (2.4), we set $E = H^{-1}(\Omega),\ E_1 = H^2(\Omega)$ and we consider the set

$$X = \overline{\bigcup_{t \geqslant t_2} S(t)B}^{H^{-1}(\Omega)},$$

where B is defined in (2.85). X is thus compact in $H^{-1}(\Omega)$, bounded in $H^2(\Omega)$ and positively invariant by $S_\varepsilon(t)$.

Let u_1 and u_2 be two solutions of (2.4) with initial data $u_{01},\ u_{02} \in X$, respectively. We set $w = u_1 - u_2$. Then, w verifies

$$\begin{cases} A_\varepsilon^{-1}\partial_t w - \Delta w + l(t)w = 0, \\ w(0) = u_1(0) - u_2(0), \end{cases} \tag{2.86}$$

where $l(t) = \displaystyle\int_0^1 f'(su_1(t) + (1 - s)u_2(t))ds$.
The next lemma gives the $H^{-1}(\Omega) \to H$-smoothing property for the difference of two solutions.

Lemma 2.6.2. *Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.4) such that $\|\Delta u_i(0)\| \leqslant \rho,\ i = 1, 2$. Then, the following estimate is valid:*

$$t\|u_1(t) - u_2(t)\|^2 + \int_t^{t+1} \|\Delta u_1(s) - \Delta u_2(s)\|^2 \mathrm{d}s \leqslant R_\rho e^{\alpha t}\|u_1(0) - u_2(0)\|_{-1}^2,\ t > 0, \tag{2.87}$$

where the constants $R_\rho,\ \alpha > 0$ are independent of ε and R_ρ depends on ρ.

Proof:
Multiplying (2.86) by $-t\Delta w$ and integrating over Ω, we find

$$\frac{1}{2}t\frac{d}{dt}(-\Delta A_\varepsilon^{-1}w, w) + t\|\Delta w\|^2 - t\,(l(t)w, \Delta w) = 0. \tag{2.88}$$

Setting $(-\Delta A_\varepsilon^{-1}w, w) = \|w\|_*^2$, where $\|.\|_* \sim \|.\|_{L^2(\Omega)}$, equation (2.88) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(t\|w\|_*^2\right) + t\|\Delta w\|^2 \leqslant& \frac{1}{2}\|w\|_*^2 + t\,(l(t)w, \Delta w)\\ \leqslant& \frac{1}{2}\|w\|_*^2 + t\|l(t)\|_{L^\infty(\Omega)}\|w\|\|\Delta w\|\\ \leqslant& \frac{1}{2}\|w\|_*^2 + tc_\rho\|w\|\|\Delta w\|\\ \leqslant& \frac{1}{2}\|w\|_*^2 + \frac{t}{2}c_\rho^2\|w\|^2 + \frac{t}{2}\|\Delta w\|^2\\ \leqslant& \frac{1}{2}\|w\|_*^2 + \frac{t}{2}c_\rho^2\|\nabla w\|^2 + \frac{t}{2}\|\Delta w\|^2,\end{aligned} \tag{2.89}$$

where $\|l(t)\|_{L^\infty(\Omega)} \leqslant C(\|u_1(0)\|_{H^2(\Omega)}, \|u_2(0)\|_{H^2(\Omega)}) \leqslant c_\rho$ and c_ρ is a positive constant depending on ρ. Integrating with respect to $s \in [0, t]$, we obtain

$$\begin{aligned}t\|w(t)\|_*^2 + \int_0^t s\|\Delta w(s)\|^2\mathrm{d}s \leqslant& \int_0^t \|w(s)\|_*^2 ds + c_\rho^2\int_0^t s\|\nabla w(s)\|^2 ds\\ \leqslant& c\int_0^t \|\nabla w(s)\|^2 ds + c_\rho^2\int_0^t s\|\nabla w(s)\|^2 ds\\ \leqslant& R_\rho e^{\alpha t}\|u_1(0) - u_2(0)\|_{-1}^2,\end{aligned} \tag{2.90}$$

where we have used the fact that $\|w\|_* \leqslant c\|\nabla w\|$ and estimates (2.8) and (2.17). Now, multiplying (2.86) by $-\Delta w$ and integrating over Ω, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|w\|_*^2\right) + \|\Delta w\|^2 \leqslant& (l(t)w, \Delta w)\\ \leqslant& \frac{c_\rho^2}{2}\|\nabla w\|^2 + \frac{1}{2}\|\Delta w\|^2.\end{aligned} \tag{2.91}$$

Integrating (2.91) over $(t, t+1)$, $t > 0$, we obtain

$$\int_t^{t+1} \|\Delta w(s)\|^2\mathrm{d}s \leqslant \|u_1(0) - u_2(0)\|_{-1}^2 R_\rho e^{\alpha t}. \tag{2.92}$$

Thus, the lemma is an immediate consequence of estimates (2.90) and (2.92). □

Lemma 2.6.3. *Let the assumptions of Lemma 2.6.2 hold. Then, the following estimate holds:*

$$\|u_1(t)-u_2(t)\|_{H^1(\Omega)}^2 + \int_t^{t+1} \|\partial_t u_1(s)-\partial_t u_2(s)\|_{-1}^2\mathrm{d}s \leqslant R_\rho e^{\alpha_\rho t}\|u_1(0)-u_2(0)\|_{-1}^2,\ t \geqslant 2, \tag{2.93}$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof:
We have

$$\begin{aligned}\|\nabla(l(t)w)\| &\leqslant \|l(t)\|_{L^\infty(\Omega)}\|\nabla w\| + c\|\nabla l(t)\|_{L^4(\Omega)}\|w\|_{L^4(\Omega)}\\ &\leqslant c(\|l(t)\|_{L^\infty(\Omega)} + \|\nabla l(t)\|_{L^4(\Omega)})\|\nabla w\| \ (n \leqslant 3).\end{aligned}$$

Using the embeddings $W \subset C(\overline{\Omega})$ and $W \subset W^{1,4}(\Omega)$, we find

$$\|\nabla l(t)\|_{L^4(\Omega)} \leqslant Q(\max_{i=1,2}\|u_i(t)\|_{L^\infty(\Omega)})\max_{i=1,2}\|u_i(t)\|_{W^{1,4}(\Omega)} \leqslant C_\rho,$$

where Q is an appropriate function and C_ρ is a positive constant depending on ρ which are independent of ε. Consequently, we obtain

$$\|\nabla(l(t)w)\| \leqslant c_\rho\|\nabla w\|,$$

where $c_\rho > 0$ depend on ρ and independent of ε.
Now, multiplying (2.86) by $(t - T_*)\partial_t w$, where $T_* \geqslant 1$ is arbitrary and such that $t - T_* \geqslant 0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}((t-T_*)\|\nabla w\|^2) + c(t-T_*)\|\partial_t w\|_{-1}^2 &= \frac{1}{2}\|\nabla w\|^2 - (t-T_*)(l(t)w, \partial_t w)\\ &\leqslant \frac{1}{2}\|\nabla w\|^2 + (t-T_*)\|\nabla(l(t)w)\|\|\partial_t w\|_{-1}\\ &\leqslant \frac{1}{2}\|\nabla w\|^2 + (t-T_*)c_\rho\|\nabla w\|\|\partial_t w\|_{-1}\\ &\leqslant (1 + c_\rho(t-T_*))\frac{\|\nabla w\|^2}{2} + \frac{c}{2}(t-T_*)\|\partial_t w\|_{-1}^2.\end{aligned} \tag{2.94}$$

Integrating (2.94) with respect to $t \in [T_*, T_* + s]$, for $T_* \geqslant 1$ and $s \in [0, 2]$, and using estimate (2.17), we find

$$s\|\nabla w(T_* + s)\|^2 + c\int_{T_*}^{T_*+s}(t-T_*)\|\partial_t w(t)\|_{-1}^2 dt \leqslant \|u_1(0) - u_2(0)\|_{-1}^2 R_\rho e^{\alpha_\rho T_*}. \tag{2.95}$$

Setting $T_* := t - 1 \geqslant 1$ and $s = 1$, estimate (2.95) yields

$$\begin{aligned}\|\nabla w(t)\|^2 + c\int_{t-1}^{t}\|\partial_t w(s)\|_{-1}^2 \mathrm{d}s &\leqslant \|u_1(0) - u_2(0)\|_{-1}^2 R_\rho e^{\alpha_\rho(t-1)}\\ &\leqslant \|u_1(0) - u_2(0)\|_{-1}^2 R_\rho e^{\alpha_\rho t},\end{aligned} \tag{2.96}$$

and the lemma is thus proven. □
We now have a $H^{-1}(\Omega) \to H^2(\Omega)-$smoothing property.

Lemma 2.6.4. *Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.4) such that $\|\Delta u_i(0)\| \leqslant \rho$, $i = 1, 2$. Then, the following estimate is valid:*

$$\|u_1(t) - u_2(t)\|_{H^2(\Omega)}^2 + \|\partial_t u_1(t) - \partial_t u_2(t)\|_{-1}^2 \leqslant R_\rho e^{\alpha_\rho t}\|u_1(0) - u_2(0)\|_{-1}^2, \ t \geqslant 3, \tag{2.97}$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof:
We differentiate equation (2.86) with respect to t and set $\theta(t) = \partial_t w(t)$. This function satisfies the equation

$$A_\varepsilon^{-1}\partial_t\theta - \Delta\theta + l(t)\theta + \partial_t l(t)w = 0,\ \theta|_{\partial\Omega} = 0.$$

Multiplying this equation by $(t-2)\theta(t)$, integrating over Ω and noting that

$$\begin{aligned}\|\partial_t l(t)w(t)\|^2 \leqslant& \max_{i=1,2}(\|u_i(t)\|_{L^\infty(\Omega)})(\|\partial_t u_1(t)\|^2_{H^1(\Omega)} + \|\partial_t u_2(t)\|^2_{H^1(\Omega)})\|\nabla w(t)\|^2\\ \leqslant& c_\rho\|\nabla w(t)\|^2 \text{ (using Lemma 2.4.9)},\end{aligned}$$

we obtain the following estimate:

$$\frac{d}{dt}((t-2)\|A_\varepsilon^{-1/2}\theta\|^2) + (t-2)\|\nabla\theta\|^2 \leqslant c_\rho(t-1)\|A_\varepsilon^{-1/2}\theta\|^2 + c_\rho(t-2)\|\nabla w\|^2,\ t \geqslant 3.$$

Applying Gronwall's Lemma, using the bound for $\|\nabla w\|^2$ and estimate (2.93), we prove that

$$\|\theta(t)\|^2_{-1} + \int_t^{t+1}\|\nabla\theta(s)\|^2\mathrm{d}s \leqslant R_\rho e^{\alpha_\rho t}\|u_1(0) - u_2(0)\|^2_{-1}, \forall t \geqslant 3.$$

Now, interpreting equation (2.86) as an elliptic equation,

$$\Delta w - l(t)w = A_\varepsilon^{-1}\partial_t w,$$

we have

$$\begin{aligned}\|\Delta w(t)\|^2 &= (A_\varepsilon^{-1}\partial_t w(t), \Delta w(t)) + (l(t)w(t), \Delta w(t))\\ &\leqslant c\|\partial_t w(t)\|^2_{-1} + \frac{1}{2}\|\Delta w(t)\|^2 + \|\nabla(l(t)w(t))\|\|\nabla w(t)\|\\ &\leqslant c\|\partial_t w(t)\|^2_{-1} + \frac{1}{2}\|\Delta w(t)\|^2 + c_\rho\|\nabla w(t)\|^2.\end{aligned}$$

Using the above estimates and (2.93), we find

$$\|\Delta w(t)\|^2 \leqslant R_\rho e^{\alpha_\rho t}\|u_1(0) - u_2(0)\|^2_{-1}, \forall t \geqslant 3.$$

□

Theorem 2.6.5. *Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.4) such that $\|\Delta u_i(0)\| \leqslant \rho$, $i = 1, 2$. Then, the following estimate is valid:*

$$\|u_1(t_2) - u_2(t_2)\|^2_{H^2(\Omega)} \leqslant R_\rho\|u_1(0) - u_2(0)\|^2_{-1}, \tag{2.98}$$

where R_ρ is a positive constant depending on ρ and independent of ε.

Proof:
Having estimate (2.97) for every $t \geqslant 3$, we rescale the time ($t \to \alpha t$) and we deduce that estimate (2.98) is valid for every $t \geqslant T'$, $T' > 0$ arbitrary. In particular, this holds for $t = t_2$. □

Lemma 2.6.6. *The semigroup $S_\varepsilon(t)$ is uniformly Hölder continuous on $[0,T]\times B$, having endowed B with the $H^{-1}-$topology, i.e.,*

$$\|A_\varepsilon^{-1/2}(S_\varepsilon(t_1)u_{01}-S_\varepsilon(t_2)u_{02})\|\leqslant R_{\rho,T}\left(\|A_\varepsilon^{-1/2}(u_{01}-u_{02})\|+|t_1-t_2|^{1/2}\right),$$

where $u_{0i}\in B$, $t_i\leqslant T, i=1,2$, and $R_{\rho,T}$ is a positive constant depending on ρ and T and independent of ε.

Proof:
The Lipschitz continuity with respect to the initial conditions is an immediate consequence of estimate (2.17). In order to verify the Hölder continuity with respect to t, we integrate (2.79) from t_1 to t_2 and find

$$\begin{aligned}\int_{t_1}^{t_2}\|A_\varepsilon^{-1/2}\partial_t u(s)\|^2ds+&\frac{1}{2}\|\nabla u(t_2)\|^2+\int_\Omega F(u(t_2))\mathrm{d}x\\ =&\frac{1}{2}\|\nabla u(t_1)\|^2+\int_\Omega F(u(t_1))\mathrm{d}x.\end{aligned}$$

Using (2.3), we have

$$\begin{aligned}\int_{t_1}^{t_2}\|A_\varepsilon^{-1/2}\partial_t u(s)\|^2ds+&\frac{1}{2}\|\nabla u(t_2)\|^2+\frac{b_{2p}}{2}\int_\Omega u^{2p}(t_2)\mathrm{d}x\\ &\leqslant 2c|\Omega|+\frac{1}{2}\|\nabla u(t_1)\|^2+\frac{3}{2}b_{2p}\int_\Omega u^{2p}(t_1)\mathrm{d}x\\ &\leqslant c_\rho \text{ (using (2.80)).}\end{aligned}$$

Consequently, we deduce that

$$\begin{aligned}\|A_\varepsilon^{-1/2}(u(t_1)-u(t_2))\|\leqslant c\|u(t_1)-u(t_2)\|_{-1}=&c\left\|\int_{t_1}^{t_2}\partial_t u(s)ds\right\|_{H^{-1}(\Omega)}\\ \leqslant&c\int_{t_1}^{t_2}\|\partial_t u(s)\|_{H^{-1}(\Omega)}\mathrm{d}s\\ \leqslant&c|t_1-t_2|^{1/2}\left(\int_{t_1}^{t_2}\|\partial_t u(s)\|^2_{H^{-1}(\Omega)}\mathrm{d}s\right)^{1/2}\\ \leqslant&c|t_1-t_2|^{1/2}\left(\int_{t_1}^{t_2}\|A_\varepsilon^{-1/2}\partial_t u(s)\|^2\mathrm{d}s\right)^{1/2}\\ \leqslant&c_\rho|t_1-t_2|^{1/2},\end{aligned}$$

where c_ρ is a positive constant depending on ρ. We have thus proved the Hölder continuity with respect to t.

□

Hence, we deduce the existence of an exponential attractor for problem (2.4).

2.7 Construction of a robust family of exponential attractors

In this section, we will apply the following theorem to the study of the long time behavior of the solutions of (2.4).

Theorem 2.7.1. *(See [11]) Let E_1 and E be two Banach spaces such that the inclusion $E_1 \subset E$ is compact and let B be a bounded set of E. We assume that there exists a family of operators $S_\varepsilon : B \to B, \varepsilon \in [0,1]$, which satisfies the following assumptions:*

1. *For every $x_1, x_2 \in B$, the following estimate is valid:*

$$\|S_\varepsilon x_1 - S_\varepsilon x_2\|_{E_1} \leqslant L\|x_1 - x_2\|_E, \tag{2.99}$$

   where the constant L is independent of ε.

2. *For every $\varepsilon \in [0,1]$, for $i \in \mathbb{N}$ and for $x \in B$, we have the estimate*

$$\|S_\varepsilon^i x - S_0^i x\|_E \leqslant K^i \varepsilon. \tag{2.100}$$

Then, for every $\varepsilon \in [0, \varepsilon_0]$, there exists an exponential attractor $\mathcal{M}_\varepsilon$ for the map S_ε in B. Moreover, the exponential attractor $\mathcal{M}_\varepsilon$ can be chosen such that the following estimate is valid:

$$dist_{sym}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leqslant C_1 \varepsilon^\nu,$$

where the constant C_1 and the exponent $0 < \nu < 1$ can be calculated explicitly and $dist_{sym}$ denotes the symmetric Hausdorff distance between two sets in E. Finally, the fractal dimension of the exponential attractors considered above is uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$,

$$\dim_F(\mathcal{M}_\varepsilon, E) \leqslant C = C(L),$$

where the constant C is independent of ε and can be calculated explicitly.

In order to apply the theorem above, let $u^0(t)$ and $u^\varepsilon(t)$ be two solutions of (2.1) with respectively zero ($\varepsilon = 0$) and nonzero ($0 < \varepsilon \leqslant 1$) parameters,

$$\partial_t u^\varepsilon + A^2 u^\varepsilon + A f(u^\varepsilon) + \varepsilon A u^\varepsilon + \varepsilon f(u^\varepsilon) = 0$$

and

$$\partial_t u^0 + A^2 u^0 + A f(u^0) = 0,$$

where $A = -\Delta$.
We have the following result on the difference of the two solutions.

Theorem 2.7.2. *Let $\|u^0(0)\|_{H^2(\Omega)}$ and $\|u^\varepsilon(0)\|_{H^2(\Omega)} \leqslant \rho$, with ρ a positive constant. Then, the following estimate holds:*

$$\|u^0(t) - u^\varepsilon(t)\|_{-1}^2 \leqslant R_\rho e^{\alpha t}(\|u^0(0) - u^\varepsilon(0)\|_{-1}^2 + \varepsilon^2), \tag{2.101}$$

where $t \geqslant 0$, α is a positive constant independent of ε and R_ρ is a positive constant depending on ρ and independent of ε.

Proof:
Let $w^\varepsilon(t) = u^\varepsilon(t) - u^0(t)$. Then, w^ε satisfies the following equation:

$$A^{-1}\partial_t w^\varepsilon + Aw^\varepsilon + l_\varepsilon(t)w^\varepsilon + \varepsilon u^\varepsilon + \varepsilon A^{-1} f(u^\varepsilon) = 0, \tag{2.102}$$

where $l_\varepsilon(t) := \int_0^1 f'(su^\varepsilon(t) + (1-s)u^0(t))\mathrm{d}s$. Multiplying (2.102) by $w^\varepsilon(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w^\varepsilon\|_{-1}^2 + \|\nabla w^\varepsilon\|^2 + (l_\varepsilon(t)w^\varepsilon, w^\varepsilon) + \varepsilon(u^\varepsilon, w^\varepsilon) + \varepsilon(A^{-1}f(u^\varepsilon), w^\varepsilon) = 0. \tag{2.103}$$

We have

$$\begin{aligned}(l_\varepsilon(t)w^\varepsilon, w^\varepsilon) &\geqslant -\kappa\|w^\varepsilon\|^2 \\ &\geqslant -\frac{1}{4}\|\nabla w^\varepsilon\|^2 - c\|w^\varepsilon\|_{-1}^2,\end{aligned}$$

$$\begin{aligned}\varepsilon(u^\varepsilon, w^\varepsilon) &\leqslant \|w^\varepsilon\|^2 + \frac{\varepsilon^2}{4}\|u^\varepsilon\|^2 \\ &\leqslant \frac{1}{4}\|\nabla w^\varepsilon\|^2 + c\|w^\varepsilon\|_{-1}^2 + \frac{\varepsilon^2}{4}\|u^\varepsilon\|^2\end{aligned}$$

and

$$\begin{aligned}\varepsilon(A^{-1}f(u^\varepsilon), w^\varepsilon) &\leqslant \varepsilon\|f(u^\varepsilon)\|_{-1}\|w^\varepsilon\|_{-1} \\ &\leqslant \|w^\varepsilon\|_{-1}^2 + \frac{\varepsilon^2}{4}\|f(u^\varepsilon)\|_{-1}^2 \\ &\leqslant \|w^\varepsilon\|_{-1}^2 + \varepsilon^2 Q(\|u^\varepsilon\|_{H^2(\Omega)}) \text{ (using (2.43))} \\ &\leqslant \|w^\varepsilon\|_{-1}^2 + c_\rho\varepsilon^2 \text{ (using (2.40))},\end{aligned}$$

where c_ρ depends on ρ and is independent of ε. Then, (2.103) gives

$$\frac{d}{dt}\|w^\varepsilon\|_{-1}^2 + \|\nabla w^\varepsilon\|^2 \leqslant c\|w^\varepsilon\|_{-1}^2 + c'\varepsilon^2 h_\varepsilon(t), \tag{2.104}$$

where $h_\varepsilon(t) := \|u^\varepsilon(t)\|^2 + c_\rho$. Applying Gronwall's lemma and using estimate (2.40), inequality (2.104) gives

$$\|w^\varepsilon(t)\|_{-1}^2 + \int_t^{t+1} \|\nabla w^\varepsilon(s)\|^2\mathrm{d}s \leqslant R_\rho e^{\alpha t}(\|w^\varepsilon(0)\|_{-1}^2 + \varepsilon^2). \tag{2.105}$$

□

Theorem 2.7.3. *Let $\|u^0(0)\|_{H^2(\Omega)}$ and $\|u^\varepsilon(0)\|_{H^2(\Omega)} \leqslant \rho$. We have the following:*

$$t\|u^\varepsilon(t) - u^0(t)\|^2 \leqslant R_\rho e^{\alpha_\rho t}(\|u^\varepsilon(0) - u^0(0)\|_{-1}^2 + \varepsilon^2),\ t \geqslant 0, \tag{2.106}$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof:
Multiplying (2.102) by tAw^ε and integrating over Ω, where $w^\varepsilon = u^\varepsilon - u^0$, we obtain

$$\frac{t}{2}\frac{d}{dt}\|w^\varepsilon\|^2 + t\|Aw^\varepsilon\|^2 + t(l_\varepsilon(t)w^\varepsilon, Aw^\varepsilon) + \varepsilon t(u^\varepsilon, Aw^\varepsilon) + \varepsilon t(f(u^\varepsilon), w^\varepsilon) = 0.$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(t\|w^\varepsilon\|^2) + t\|\Delta w^\varepsilon\|^2 =& t(l_\varepsilon(t)w^\varepsilon, \Delta w^\varepsilon) + \varepsilon t(u^\varepsilon, \Delta w^\varepsilon) - \varepsilon t(f(u^\varepsilon), w^\varepsilon) + \frac{1}{2}\|w^\varepsilon\|^2\\ \leqslant& t\|l_\varepsilon(t)\|_\infty\|w^\varepsilon\|\|\Delta w^\varepsilon\| + \varepsilon t\|u^\varepsilon\|\|\Delta w^\varepsilon\| + \varepsilon t\|f(u^\varepsilon)\|\|w^\varepsilon\| + \frac{1}{2}\|w^\varepsilon\|^2\\ \leqslant& tc_\rho^2\|w^\varepsilon\|^2 + \frac{t}{4}\|\Delta w^\varepsilon\|^2 + \varepsilon^2 t\|u^\varepsilon\|^2 + \frac{t}{4}\|\Delta w^\varepsilon\|^2 + \varepsilon^2 t\|f(u^\varepsilon)\|^2\\ &+ t\|w^\varepsilon\|^2 + \frac{1}{2}\|w^\varepsilon\|^2,\end{aligned}$$

where $\|l_\varepsilon(t)\|_{L^\infty(\Omega)} \leqslant Q(\max(\|u^\varepsilon(t)\|_{L^\infty(\Omega)}, \|u^0(t)\|_{L^\infty(\Omega)})) \leqslant c_\rho$. Therefore, we have

$$\begin{aligned}\frac{d}{dt}(t\|w^\varepsilon\|^2) + t\|\Delta w^\varepsilon\|^2 \leqslant& (c_\rho^2+1)t\|w^\varepsilon\|^2 + \varepsilon^2 t(\|u^\varepsilon\|^2 + \|f(u^\varepsilon)\|^2) + \|w^\varepsilon\|^2\\ \leqslant& (c_\rho^2+1)t\|w^\varepsilon\|^2 + \varepsilon^2 t(\|u^\varepsilon\|^2 + c_\rho) + c\|\nabla w^\varepsilon\|^2,\end{aligned}$$

where we have used (2.44). Applying Gronwall's lemma and using (2.40) and estimate (2.105), we deduce that

$$t\|u^\varepsilon(t) - u^0(t)\|^2 + \int_t^{t+1} s\|\Delta w^\varepsilon(s)\|^2 \mathrm{d}s \leqslant R_\rho e^{\alpha_\rho t}(\|u^\varepsilon(0) - u^0(0)\|_{-1}^2 + \varepsilon^2). \quad (2.107)$$

□

Theorem 2.7.4. *Let the assumptions of Theorem 2.7.2 hold. Then, the following estimate is valid:*

$$t\|u^\varepsilon(t)-u^0(t)\|_{H^1(\Omega)}^2+\int_t^{t+1} s\|\partial_t u^\varepsilon(s)-\partial_t u^0(s)\|_{-1}^2 \mathrm{d}s \leqslant R_\rho e^{\alpha_\rho t}(\|u^\varepsilon(0)-u^0(0)\|_{-1}^2+\varepsilon^2),\ t>0, \quad (2.108)$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof:
Multiplying (2.102) by $t\partial_t w^\varepsilon$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(t\|\nabla w^\varepsilon\|^2) + t\|\partial_t w^\varepsilon\|_{-1}^2 =& \frac{1}{2}\|\nabla w^\varepsilon\|^2 - t(l_\varepsilon(t)w^\varepsilon, \partial_t w^\varepsilon) - \varepsilon t(u^\varepsilon, \partial_t w^\varepsilon) - \varepsilon t(A^{-1}f(u^\varepsilon), \partial_t w^\varepsilon)\\ \leqslant& \frac{1}{2}\|\nabla w^\varepsilon\|^2 + t\|\nabla(l_\varepsilon(t)w^\varepsilon)\|\|\partial_t w^\varepsilon\|_{-1} + \varepsilon t\|\nabla u^\varepsilon\|\|\partial_t w^\varepsilon\|_{-1}\\ &+ \varepsilon t\|f(u^\varepsilon)\|_{-1}\|\partial_t w^\varepsilon\|_{-1}\\ \leqslant& \frac{1}{2}\|\nabla w^\varepsilon\|^2 + c_\rho t\|\nabla w^\varepsilon\|^2 + \frac{t}{2}\|\partial_t w^\varepsilon\|_{-1}^2 + c_\rho\varepsilon^2 t(\|\nabla u^\varepsilon\|^2 + 1).\end{aligned}$$

Applying Gronwall's lemma, using estimates (2.105), (2.107) and Theorem 2.4.1, we deduce estimate (2.108). □

Theorem 2.7.5. *Let $\|u^0(0)\|_{H^2(\Omega)}$ and $\|u^\varepsilon(0)\|_{H^2(\Omega)} \leqslant \rho$. We have the following:*

$$t\|u^\varepsilon(t)-u^0(t)\|^2_{H^2(\Omega)}+t\|\partial_t u^\varepsilon(t)-\partial_t u^0(t)\|^2_{-1} \leqslant R_\rho e^{\alpha_\rho t}(\|u^\varepsilon(0)-u^0(0)\|^2_{-1}+\varepsilon^2),\ t \geqslant 0, \tag{2.109}$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof:
Differentiating (2.102) with respect to t and setting $\theta^\varepsilon = \partial_t w^\varepsilon$, we obtain

$$A^{-1}\partial_t\theta^\varepsilon + A\theta^\varepsilon + l_\varepsilon(t)\theta^\varepsilon + \partial_t l_\varepsilon(t)w^\varepsilon + \varepsilon\partial_t u^\varepsilon + \varepsilon A^{-1}f'(u^\varepsilon(t))\partial_t u^\varepsilon = 0. \tag{2.110}$$

Multiplying (2.110) by $t\theta^\varepsilon$, using the fact that $l_\varepsilon(t) \geqslant -\kappa$, $\|\partial_t l_\varepsilon(t)w^\varepsilon\| \leqslant c_\rho\|\nabla w^\varepsilon\|$ and
$\|f'(u^\varepsilon)\|_{L^\infty(\Omega)} \leqslant c_\rho$, we find

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(t\|\theta^\varepsilon\|^2_{-1}) + t\|\nabla\theta^\varepsilon\|^2 \leqslant & \frac{1}{2}\|\theta^\varepsilon\|^2_{-1} + \kappa t\|\theta^\varepsilon\|^2 - t(\partial_t l_\varepsilon(t)w^\varepsilon, \theta^\varepsilon) \\
& - \varepsilon t(\partial_t u^\varepsilon, \theta^\varepsilon) - \varepsilon t(A^{-1}f'(u^\varepsilon)\partial_t u^\varepsilon, \theta^\varepsilon) \\
\leqslant & \frac{1}{2}\|\theta^\varepsilon\|^2_{-1} + ct\|\theta^\varepsilon\|_{-1}\|\nabla\theta^\varepsilon\| + t\|\partial_t l_\varepsilon(t))w^\varepsilon\|\|\theta^\varepsilon\| + \varepsilon t\|\partial_t u^\varepsilon\|_{-1}\|\nabla\theta^\varepsilon\| \\
& + \varepsilon t\|f'(u^\varepsilon)\partial_t u^\varepsilon\|_{-1}\|\theta^\varepsilon\|_{-1} \\
\leqslant & c_\rho(t+1)\|\theta^\varepsilon\|^2_{-1} + c_\rho t(\|\nabla w^\varepsilon\|^2 + \varepsilon^2\|\partial_t u^\varepsilon\|^2_{-1}) + \frac{t}{2}\|\nabla\theta^\varepsilon\|^2.
\end{aligned}$$

Using estimates (2.48) and (2.108), we deduce that

$$t\|\partial_t u^\varepsilon(t) - \partial_t u^0(t)\|^2_{-1} \leqslant R_\rho e^{\alpha_\rho t}(\|u^\varepsilon(0) - u^0(0)\|^2_{-1} + \varepsilon^2),\ t \geqslant 0. \tag{2.111}$$

Interpreting equation (2.102) as an elliptic equation,

$$Aw^\varepsilon(t) + l_\varepsilon(t)w^\varepsilon(t) = -A^{-1}\partial_t w^\varepsilon(t) - \varepsilon u^\varepsilon(t) - \varepsilon A^{-1}f(u^\varepsilon(t)) \tag{2.112}$$

and multiplying (2.112) by $Aw^\varepsilon(t)$, we obtain

$$\begin{aligned}
\|Aw^\varepsilon(t)\|^2 = & -(l_\varepsilon(t)w^\varepsilon(t), Aw^\varepsilon(t)) - (A^{-1}\partial_t w^\varepsilon(t), Aw^\varepsilon(t)) - (\varepsilon u^\varepsilon(t), Aw^\varepsilon(t)) \\
& - (\varepsilon A^{-1}f(u^\varepsilon(t)), Aw^\varepsilon(t)) \\
\leqslant & c_\rho\|\nabla w^\varepsilon(t)\|\|Aw^\varepsilon(t)\| + \|\partial_t w^\varepsilon(t)\|_{-1}\|\nabla w^\varepsilon(t)\| + \varepsilon\|u^\varepsilon(t)\|\|Aw^\varepsilon(t)\| \\
& + \varepsilon\|f(u^\varepsilon(t))\|_{-1}\|\nabla w^\varepsilon(t)\|,
\end{aligned}$$

hence

$$\|Aw^\varepsilon(t)\|^2 \leqslant c_\rho(\|\partial_t w^\varepsilon(t)\|^2_{-1} + \|\nabla w^\varepsilon(t)\|^2 + \varepsilon^2). \tag{2.113}$$

Using (2.111) and (2.113), we have the desired inequality. □

Theorem 2.7.6. *Let the above assumptions hold. Then, for every $\varepsilon \geqslant 0$, the semigroup $S_\varepsilon(t)$ generated by equation (2.4) possesses an exponential attractor $\mathcal{M}_\varepsilon \subset W$. Moreover, these exponential attractors can be chosen such that*

$$\dim_F(\mathcal{M}_\varepsilon, W) \leqslant C, \tag{2.114}$$

$$dist_{sym,W}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leqslant C'\varepsilon^\eta, \tag{2.115}$$

where the constants $C, C' > 0$ and $0 < \eta < 1$ are independent of ε and can be calculated explicitly and where the rate of convergence to these attractors is also uniform with respect to ε, i.e., there exists a constant $\alpha > 0$ such that, for every bounded subset $B_0 \subset W$, there exists a constant $C'' = C''(B_0)$ such that

$$dist_W(S_\varepsilon(t)B_0, \mathcal{M}_\varepsilon) \leqslant C''e^{-\alpha t},$$

where C'' and α are also independent of ε

Proof:
We recall that the set

$$B = \{u_0 \in W, \|\Delta u_0\| \leqslant \rho\}$$

is a uniformly absorbing set for the semigroup $S_\varepsilon(t), \varepsilon \in [0,1]$, i.e., for every bounded set $B_0 \subset H^2(\Omega)$, there exists a time T which is independent of ε, such that

$$S_\varepsilon(t)B_0 \subset B,\ \forall t \geqslant T,\ \forall \varepsilon \in [0,1].$$

Therefore, it is sufficient to construct exponential attractors $\mathcal{M}_\varepsilon$ on B only. We define a family of maps $S_\varepsilon := S_\varepsilon(t_2) : B \to B$, for $\varepsilon \in [0,1]$ and then construct exponential attractors $\mathcal{M}_\varepsilon^d$ for the discrete semigroups generated by these maps. Moreover, it is convenient to construct these attractors on B endowed first with the metric of the space $H^{-1}(\Omega)$. Then, we apply Theorem 2.7.1 with $E := H^{-1}(\Omega)$ and $E_1 := H^2(\Omega)$. Estimate (2.99) is an immediate consequence of Theorem 2.6.5 and estimate (2.100) is a corollary of Theorem 2.7.2. Thus, all assumptions of Theorem 2.7.1 are satisfied and, consequently, we have

$$\begin{aligned}
\dim_F(\mathcal{M}_\varepsilon^d, H^{-1}(\Omega)) &\leqslant C,\\
\operatorname{dist}_{sym,H^{-1}(\Omega)}(\mathcal{M}_\varepsilon^d, \mathcal{M}_0^d) &\leqslant C'\varepsilon^\nu,\\
\operatorname{dist}_{H^{-1}(\Omega)}(S_\varepsilon^n B, \mathcal{M}_\varepsilon^d) &\leqslant C''e^{-\alpha n},
\end{aligned}$$

for appropriate constants C, C', C'', α and ν which are independent of ε. Finally, we set

$$\mathcal{M}_\varepsilon^c := \cup_{t\in[0,t_2]} S_\varepsilon(t)\mathcal{M}_\varepsilon^d.$$

Since, $(t,x) \to S_\varepsilon(t)x$ is uniformly Hölder continuous (Lemma 2.6.6) on $[0,t_2] \times B$, the exponential attractors $\mathcal{M}_\varepsilon^c,\ \varepsilon \in [0,1]$, satisfy

1.
$$\dim_F(\mathcal{M}_\varepsilon^c, H^{-1}(\Omega)) \leqslant C + 2, \tag{2.116}$$

2.
$$\operatorname{dist}_{H^{-1}(\Omega)}(S_\varepsilon(t)B, \mathcal{M}_\varepsilon^d) \leqslant C_1 e^{-\alpha t/t_2}, \tag{2.117}$$

3.
$$\operatorname{dist}_{sym,H^{-1}(\Omega)}(\mathcal{M}_\varepsilon^c, \mathcal{M}_0^c) \leqslant C\left(\operatorname{dist}_{sym,H^{-1}(\Omega)}(\mathcal{M}_\varepsilon^d, \mathcal{M}_0^d) + \varepsilon\right) \leqslant C_2\varepsilon^\nu. \tag{2.118}$$

Thus, we have constructed exponential attractors $\mathcal{M}_\varepsilon^c$ for the metric of $H^{-1}(\Omega)$. We note that it follows from its definition that the set $S_\varepsilon(1)\mathcal{M}_\varepsilon^c$ is also an exponential attractor. We finally set

$$\mathcal{M}_\varepsilon := S_\varepsilon(1)\mathcal{M}_\varepsilon^c.$$

Thus, we deduce estimates (2.114) and (2.115) from estimates (2.116), (2.118) and from Lemma 2.6.2 and Theorem 2.7.5 for $t = 1$. □

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Chapitre 3

A Cahn-Hilliard Type Equation With Dynamic Boundary Conditions and Regular Potentials

Equation de type Cahn-Hilliard avec des conditions dynamiques sur le bord et des potentiels réguliers

3.1 Introduction

Given a bounded and smooth domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma := \partial\Omega$, we consider the following problem:

$$\begin{cases} \partial_t u = \Delta w - w = -(-\Delta + I)w = -Aw, \ \partial_n w_{|\Gamma} = 0, \\ w = -\Delta u + f(u), \ u_{|t=0} = u_0, \\ \partial_t v = \Delta_\Gamma v - g(v) - \lambda v - \partial_n u, \ x \in \Gamma, \ v_{|t=0} = v_0 \\ u_{|\Gamma} = v \end{cases} \tag{3.1}$$

where Δ_Γ is the Laplace-Beltrami operator on the boundary Γ, n is the outward normal direction to the boundary and λ is some given positive constant. The nonlinear functions f and g represent the derivative of a typically nonconvex configuration potential and of a boundary potential, respectively.
The boundary condition will be interpreted as an additional second-order parabolic equation on the boundary Γ.
We note that equation (3.1) may be viewed as a combination of the well-known Cahn-Hilliard equation

$$\partial_t u = -\Delta(\Delta u - f(u)), \quad u(0,x) = u_0(x)$$

and of the Allen-Cahn equation

$$\partial_t u = \Delta u - f(u), \quad u(0,x) = u_0(x).$$

Problem (3.1) is introduced by [30] as a simplification of a mesoscopic model for multiple microscopic mechanisms in surface processes such that surface diffusion and adsorption-desorption; u is the order parameter representing typically the normalized concentration of one component and w is the chemical potential.
This problem was derived and studied in [28], [27], [29], [31] and [32] where questions such as existence and uniqueness of solutions, existence of the global attractor and of an exponential attractor have been answered under various assumptions on the nonlinearities.
We have the following free energy for our problem:

$$\mathcal{E}(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) \mathrm{d}x + \int_\Gamma \left(\frac{1}{2}|\nabla_\Gamma u|^2 + \frac{\lambda}{2}|u|^2 + G(u)\right) \mathrm{d}\sigma$$

where $F(s) := \displaystyle\int_0^s f(t)dt$ and $G(s) := \displaystyle\int_0^s g(t)dt$. The first integral is the bulk energy and the second one is the surface energy. If u is a regular solution of problem (3.1), then, u dissipates $\mathcal{E}$ since

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\int_\Omega (|\nabla w|^2 + |w|^2)\mathrm{d}x - \int_\Gamma |u_t|^2 \mathrm{d}\sigma.$$

In what follows, unless mentioned explicitly, the same letter Q denotes monotone increasing functions and the same letter c denotes positive constants independent of t, possibly changing at different occurrences.

3.1.1 Assumptions and notations

We assume that the nonlinearities f and g belong to $C^2(\mathbb{R},\mathbb{R})$ and satisfy the following standard dissipativity assumptions

$$\liminf_{|s|\to+\infty} f'(s) > 0, \ \liminf_{|s|\to+\infty} g'(s) > 0. \tag{3.2}$$

We note that assumptions (3.2) imply that (see [42])

$$\begin{aligned} \frac{1}{2}|f(u)|(1+|u|) &\leqslant f(u)u + c \\ \frac{1}{2}|g(v)|(1+|v|) &\leqslant g(v)v + c \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} |F(s)| &\leqslant |f(s)|(1+|s|) - c \\ |G(s)| &\leqslant |g(s)|(1+|s|) - c. \end{aligned} \tag{3.4}$$

Consequently, estimates (3.3) and (3.4) yield

$$\begin{aligned} (f(u),u)_\Omega &\geqslant c(\|f(u)\|_{L^1(\Omega)} + \|F(u)\|_{L^1(\Omega)}) - c \\ (g(v),v)_\Gamma &\geqslant c(\|g(v)\|_{L^1(\Gamma)} + \|G(v)\|_{L^1(\Gamma)}) - c, \end{aligned} \tag{3.5}$$

where $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ denote the standard scalar products in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively.
Now, we introduce the following natural phase space for problem (3.1)

$$\mathbb{W} := \{(u,v) \in H^2(\Omega) \times H^2(\Gamma),\ w = -\Delta u + f(u) \in H^1(\Omega),\ u_{|\Gamma} = v,\ \partial_n w_{|\Gamma} = 0\}$$

endowed with the obvious norm

$$\|(u,v)\|^2_{\mathbb{W}} := \|u\|^2_{H^2(\Omega)} + \|w\|^2_{H^1(\Omega)} + \|v\|^2_{H^2(\Gamma)}.$$

We also need to introduce the following spaces

$$\mathbb{H} := L^2(\Omega) \times L^2(\Gamma)$$

$$\mathbb{V} := \{(u,v) \in H^1(\Omega) \times H^1(\Gamma),\ u_{|\Gamma} = v\}$$

where the scalar products are given by

$$(U,U')_{\mathbb{H}} = (u,u')_\Omega + (v,v')_\Gamma,\ \forall U = (u,v),\ U' = (u',v') \in \mathbb{H},$$
$$(U,U')_{\mathbb{V}} = (\nabla u, \nabla u')_\Omega + \lambda\,(v,v')_\Gamma + (\nabla_\Gamma v, \nabla_\Gamma v')_\Gamma,\ \forall U = (u,v),\ U' = (u',v') \in \mathbb{V},$$

respectively. We further introduce the space $\mathbb{L} = H^{-1}(\Omega) \times L^2(\Gamma)$ defined by the following norm

$$\|(u,v)\|^2_{\mathbb{L}} := \|u\|^2_{H^{-1}(\Omega)} + \|v\|^2_{L^2(\Gamma)}$$

where $H^{-1}(\Omega) = [H^1(\Omega)]'$. We endow $H^{-1}(\Omega)$ with the norm

$$\|\mu\|^2_{H^{-1}(\Omega)} := \|A^{-1/2}\mu\|^2_{L^2(\Omega)} = \left(A^{-1}\mu,\mu\right)_\Omega$$

where $A^{-1} : L^2(\Omega) \to L^2(\Omega)$ denotes the inverse of the operator $(-\Delta + I)$ with Neumann boundary conditions.

3.2 Uniform a priori estimates

In this section, the main task is to derive several estimates for the solutions of problem (3.1) which are necessary for the study of the asymptotic behavior.

Definition 3.2.1. A solution of problem (3.1) is a pair $(u(t), v(t))$ of functions belonging to the space $L^\infty([0,T],\mathbb{W})$ with $(\partial_t u, \partial_t v) \in L^2([0,T],\mathbb{V})$ and which satisfies the equations (3.1) in the sense of equalities in the spaces $L^2([0,T],L^2(\Omega))$ and $L^2([0,T],L^2(\Gamma))$.

We have the following:

Proposition 3.2.2. *Let the nonlinearities f and g satisfy* (3.2) *and let $(u(t),v(t))$ be a solution of problem* (3.1)*. Then, the following estimate holds*

$$\begin{aligned}\|(u(t),v(t))\|^2_{\mathbb{L}} + c\int_t^{t+1} (\|u(s)\|^2_{H^1(\Omega)} + \|v(s)\|^2_{H^1(\Gamma)} + \|f(u(s))\|_{L^1(\Omega)} + \|g(v(s))\|_{L^1(\Gamma)})\mathrm{d}s \\ + c\int_t^{t+1} (\|F(u(s))\|_{L^1(\Omega)} + \|G(v(s))\|_{L^1(\Gamma)})\mathrm{d}s \\ \leqslant \|(u(0),v(0))\|^2_{\mathbb{L}} e^{-kt} + c,\end{aligned} \tag{3.6}$$

where the positive constant k is independent of t.

Proof:
Multiplying the first equation of (3.1) by $A^{-1}u(t)$, the second one by $u(t)$ and the third one by $v(t)$ and taking the sum of the resulting equations that we obtain, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2_{H^{-1}(\Omega)}+\|v(t)\|^2_{L^2(\Gamma)}\right)+&\|\nabla u(t)\|^2_{L^2(\Omega)}+\|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)}+\lambda\|v\|^2_{L^2(\Gamma)}\\&+(f(u(t)),u(t))_\Omega+(g(v(t)),v(t))_\Gamma=0.\end{aligned}\tag{3.7}$$

Using (3.5), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2_{H^{-1}(\Omega)}+\|v(t)\|^2_{L^2(\Gamma)}\right)+\|\nabla u(t)\|^2_{L^2(\Omega)}+\|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)}+\lambda\|v\|^2_{L^2(\Gamma)}\\&\quad+c(\|f(u(t))\|_{L^1(\Omega)}+\|g(v(t))\|_{L^1(\Gamma)}+\|F(u(t))\|_{L^1(\Omega)}+\|G(v(t))\|_{L^1(\Gamma)})\leqslant c.\end{aligned}\tag{3.8}$$

Using the fact that

$$\|u\|^2_{H^1(\Omega)}\leqslant c_\Omega(\|\nabla u\|^2_{L^2(\Omega)}+\|v\|^2_{H^1(\Gamma)})\tag{3.9}$$

equation (3.8) gives

$$\begin{aligned}&\frac{d}{dt}(\|u(t)\|^2_{H^{-1}(\Omega)}+\|v(t)\|^2_{L^2(\Gamma)})\\&+k(\|u(t)\|^2_{H^{-1}(\Omega)}+\|v(t)\|^2_{L^2(\Gamma)})+c_k(\|u(t)\|^2_{H^1(\Omega)}+\|v(t)\|^2_{H^1(\Gamma)})\\&+c(\|f(u(t))\|_{L^1(\Omega)}+\|g(v(t))\|_{L^1(\Gamma)}+\|F(u(t))\|_{L^1(\Omega)}+\|G(v(t))\|_{L^1(\Gamma)})\leqslant c,\end{aligned}\tag{3.10}$$

where $k>0$ is small enough and $c_k>0$ depends on k and is independent of t. Consequently, applying Gronwall's inequality to (3.10) followed by an integration over the time interval $(t,t+1)$, we obtain estimate (3.6). □

Proposition 3.2.3. *Let the assumptions of Proposition 3.2.2 hold. Then, we have the following estimate:*

$$\begin{aligned}&\|u(t)\|^2_{H^1(\Omega)}+\|v(t)\|^2_{H^1(\Gamma)}+(F(u(t)),1)_\Omega+(G(u(t)),1)_\Gamma\\&+\int_0^t\left(\|\partial_t u(s)\|^2_{H^{-1}(\Omega)}+\|\partial_t v(s)\|^2_{L^2(\Gamma)}\right)e^{-\tau(t-s)}\mathrm{d}s\\&+\int_t^{t+1}\left(\|\partial_t u(s)\|^2_{H^{-1}(\Omega)}+\|\partial_t v(s)\|^2_{L^2(\Gamma)}\right)\mathrm{d}s\\\leqslant&c\left(\|u(0)\|^2_{H^1(\Omega)}+\|v(0)\|^2_{H^1(\Gamma)}+(F(u(0)),1)_\Omega+(G(u(0)),1)_\Gamma\right)e^{-\tau t}+c,\end{aligned}\tag{3.11}$$

where the positive constant τ is independent of t.

Proof:
Multiplying the first equation of (3.1) by $A^{-1}\partial_t u(t)$, the second one by $\partial_t u(t)$ and

the third one by $\partial_t v(t)$ and taking the sum of the resulting equations, we have the following identity

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)} + \lambda\|v(t)\|^2_{L^2(\Gamma)} + 2\left(F(u(t)),1\right)_\Omega + 2\left(G(v(t)),1\right)_\Gamma) + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)} = 0. \tag{3.12}$$

Summing up (3.8) and (3.12), we obtain

$$\begin{aligned}&\frac{d}{dt}E(t) + 2\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + 2\|\partial_t v(t)\|^2_{L^2(\Gamma)} \\ &\quad + 2(\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)} + \lambda\|v\|^2_{L^2(\Gamma)} + c\|F(u(t))\|_{L^1(\Omega)} + c\|G(v(t))\|_{L^1(\Gamma)}) \\ &\leqslant c,\end{aligned}$$

where

$$\begin{aligned}E(t) =&\|u(t)\|^2_{H^{-1}(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)} + \lambda\|v(t)\|^2_{L^2(\Gamma)} \\ &+ 2\left(F(u(t)),1\right)_\Omega + 2\left(G(v(t)),1\right)_\Gamma.\end{aligned}$$

Thus, for $\tau > 0$ small enough, we have

$$\frac{d}{dt}E(t) + \tau E(t) + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)} \leqslant c.$$

Applying the Gronwall lemma, we obtain (3.11). □

Proposition 3.2.4. *Let the assumptions of Proposition 3.2.2 hold. Then, we have the following estimate:*

$$\begin{aligned}\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)} + \int_0^t (\|\partial_t u(s)\|^2_{H^1(\Omega)} + \|\partial_t v(s)\|^2_{H^1(\Gamma)})e^{-\rho(t-s)}\mathrm{d}s \\ \leqslant c\|(u(0),v(0))\|^2_{\mathbb{W}}e^{-\rho t} + c,\end{aligned} \tag{3.13}$$

where the constant ρ is independent of t.

Proof:
We differentiate (3.1) with respect to t and set $(\phi(t),\mu(t),\psi(t)) := \partial_t(u(t),w(t),v(t))$. Then, we have

$$\begin{cases}\partial_t\phi = \Delta\mu - \mu = -(-\Delta + I)\mu = -A\mu,\ \partial_n\mu_{|\Gamma} = 0,\\ \mu = -\Delta\phi + f'(u)\phi,\\ \partial_t\psi = \Delta_\Gamma\psi - g'(v)\psi - \lambda\psi - \partial_n\psi,\ x\in\Gamma,\\ \phi_{|\Gamma} = \psi\end{cases} \tag{3.14}$$

Multiplying the first, second and third equations of (3.14) by $A^{-1}\phi(t)$, $\phi(t)$ and $\psi(t)$, respectively, then integrate over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}) + \|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + \lambda\|\psi(t)\|^2_{L^2(\Gamma)} \\ + (f'(u)\phi(t),\phi(t))_\Omega + (g'(v)\psi(t),\psi(t))_\Gamma = 0.\end{aligned} \tag{3.15}$$

We define $h(t) := -\left(f'(u)\phi(t), \phi(t)\right)_\Omega - \left(g'(v)\psi(t), \psi(t)\right)_\Gamma$. Due to assumption (3.1), we have $f'(s) \geqslant -c$ and $g'(s) \geqslant -c$, for all $s \in \mathbb{R}$. Thus, using the following interpolation inequality $\|\phi\|^2_{L^2(\Omega)} \leqslant c\|\phi\|_{H^{-1}(\Omega)}\|\phi\|_{H^1(\Omega)}$, $h(t)$ is estimated as follows

$$\begin{aligned} h(t) &\leqslant c\|\phi(t)\|^2_{L^2(\Omega)} + c\|\psi(t)\|^2_{L^2(\Gamma)} \\ &\leqslant \eta\|\phi(t)\|^2_{H^1(\Omega)} + c_\eta\|\phi(t)\|^2_{H^{-1}(\Omega)} + c\|\psi(t)\|^2_{L^2(\Gamma)}, \end{aligned}$$

for a sufficiently small $\eta > 0$ and a large constant $c_\eta > 0$. Choosing $\eta < 1$ small and using estimate (3.11) as well as the embeddings $H^2(\Omega) \subset L^\infty(\Omega), H^2(\Gamma) \subset L^\infty(\Gamma)$, we obtain

$$\sup_{t\geqslant 0}\int_t^{t+1} h(s)\mathrm{d}s \leqslant Q(\|(u(0), v(0))\|^2_{\mathbb{W}})e^{-\tau t} + c. \tag{3.16}$$

Thus, for a ρ small enough, (3.15) gives:

$$\begin{aligned} \frac{d}{dt}(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}) + \rho(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}) \\ + \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} \leqslant c_\eta(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}). \end{aligned} \tag{3.17}$$

Applying the Gronwall lemma and using (3.11), (3.16), we find

$$\begin{aligned} &\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} + \int_0^t (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)})e^{-\rho(t-s)}\mathrm{d}s \\ \leqslant& c(\|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)} + \|(u(0), v(0))\|^2_{\mathbb{V}} + (F(u(0)), 1)_\Omega + (G(u(0)), 1)_\Gamma)e^{-\rho t} \\ &+ c. \end{aligned}$$

Moreover, from (3.1), we have

$$\phi(0) = \Delta w(0) - w(0) = -Aw(0),$$

$$\psi(0) = \Delta_\Gamma v(0) - \partial_n u(0) - \lambda v(0) - g(v(0)).$$

Therefore, we deduce from $H^2(\Omega) \subset L^\infty(\Omega)$ and $H^2(\Gamma) \subset L^\infty(\Gamma)$ that

$$\begin{aligned} \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)} =& \|Aw(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)} \\ =& \|A^{1/2}w(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)} \\ \leqslant& c\left(\|w(0)\|^2_{H^1(\Omega)} + \|(u(0), v(0))\|^2_{H^2(\Omega)\times H^2(\Gamma)}\right) \\ =& c\|(u(0), v(0))\|^2_{\mathbb{W}}. \end{aligned}$$

Using (3.3) and the above estimates, we deduce (3.13). □

Theorem 3.2.5. *Let the assumptions of Proposition 3.2.2 hold. Then, we have the following estimate:*

$$\|(u(t), v(t))\|_{\mathbb{W}} \leqslant Q(\|(u(0), v(0))\|_{\mathbb{W}})e^{-\alpha t} + c, \tag{3.18}$$

where the positive constant α is independent of t.

Proof:
We recall that $\phi = \partial_t u = \Delta w - w = -Aw$. Therefore, (3.13) can be also rewritten as

$$\begin{aligned}\|A^{1/2}w(t)\|^2_{L^2(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} + \int_0^t (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)})e^{-\rho(t-s)}\mathrm{d}s \\ \leqslant Q(\|u(0), v(0)\|^2_{\mathbb{W}})e^{-\rho t} + c.\end{aligned} \tag{3.19}$$

In order to obtain the required estimates of u and v in the $H^2(\Omega)-$norms, we can rewrite (for every fixed t) problem (3.1) as a second order nonlinear elliptic boundary value problem. We have

$$\begin{cases} -\Delta u(t) + f(u(t)) = h_1(t) := w(t),\ u_{|\Gamma} = v \\ -\Delta_\Gamma v(t) + \lambda v(t) + g(v(t)) + \partial_n u(t) = h_2(t) := -\partial_t v(t) \end{cases} \tag{3.20}$$

According to (3.19), we find

$$\|h_1(t)\|^2_{L^2(\Omega)} + \|h_2(t)\|^2_{L^2(\Gamma)} \leqslant e^{-\rho t}Q(\|(u(0), v(0))\|^2_{\mathbb{W}}) + c. \tag{3.21}$$

Applying the maximal principle (see, Lemma A.2, [42]) to this problem, we obtain

$$\|u(t)\|^2_{L^\infty(\Omega)} + \|v(t)\|^2_{L^\infty(\Gamma)} \leqslant e^{-\rho t}Q(\|(u(0), v(0))\|^2_{\mathbb{W}}) + c.$$

Finally, applying the above estimate combined with a H^2-regularity theorem (see, Lemma A.1, [42]) to the elliptic boundary value problem (3.20), we find

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Gamma)} \leqslant Q(\|(u(0), v(0))\|^2_{\mathbb{W}})e^{-\rho t} + c,$$

where Q is a monotonic function independent of t. □
In the sequel, we shall also need the uniform bounds on the solutions in $H^3(\Omega) \times H^3(\Gamma)$ which are formulated in the next theorem.

Theorem 3.2.6. *Let the assumptions of Theorem* (3.2.5) *hold and let* $(u(t), v(t))$ *be a solution of problem* (3.1). *Then* $(u(t), v(t)) \in H^3(\Omega) \times H^3(\Gamma)$ *for every* $t > 0$ *and the following estimate holds:*

$$\|u(t)\|^2_{H^3(\Omega)} + \|v(t)\|^2_{H^3(\Gamma)} \leqslant \frac{1}{t}Q(\|(u(0), v(0))\|^2_{\mathbb{W}}), \tag{3.22}$$

for every $t \in]0, 1]$.

Proof:
We differentiate (3.1) and set $(\phi(t), \mu(t), \psi(t)) := \partial_t(u(t), w(t), v(t))$. Then, these functions satisfy (3.14). We multiply scalarly the first, second and third equations of (3.14) respectively by $tA^{-1}\partial_t\phi(t)$, $t\partial_t\phi(t)$ and $t\partial_t\psi(t)$ and take the sum of the equations that we obtain. Then, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(t\|\nabla\phi(t)\|^2_{L^2(\Omega)} + t\|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + t\lambda\|\psi(t)\|^2_{L^2(\Gamma)}) \\ &\quad + t\|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + t\|\partial_t\psi(t)\|^2_{L^2(\Gamma)} \\ =&\frac{1}{2}(\|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + \lambda\|\psi(t)\|^2_{L^2(\Gamma)}) \\ &\quad - (f'(u(t))\phi(t), t\partial_t\phi(t))_\Omega - (g'(v(t))\psi(t), t\partial_t\psi(t))_\Gamma .\end{aligned} \tag{3.23}$$

We estimate the last two terms on the right-hand side of (3.23) as follows

$$\begin{aligned}(f'(u(t))\phi(t), t\partial_t\phi(t))_\Omega &\leqslant ct\|f'(u(t))\phi(t)\|_{H^1(\Omega)}\|\partial_t\phi(t)\|_{H^{-1}(\Omega)} \text{ (see Chapter 2)}\\ &\leqslant Q(\|u(t)\|_{H^2(\Omega)})t\|\phi(t)\|^2_{H^1(\Omega)} + t/2\|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)}\\ &\leqslant Q(\|u(t)\|_{H^2(\Omega)})\|\phi(t)\|^2_{H^1(\Omega)} + t/2\|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)},\end{aligned}$$

$$\begin{aligned}(g'(v(t))\psi(t), t\partial_t\psi(t))_\Gamma &\leqslant ct\|g'(v(t))\psi(t)\|_{L^2(\Gamma)}\|\partial_t\psi(t)\|_{L^2(\Gamma)}\\ &\leqslant Q(\|v(t)\|_{H^2(\Gamma)})t\|\psi(t)\|^2_{L^2(\Gamma)} + t/2\|\partial_t\psi(t)\|^2_{L^2(\Gamma)}\\ &\leqslant Q(\|v(t)\|_{H^2(\Gamma)})\|\psi(t)\|^2_{L^2(\Gamma)} + t/2\|\partial_t\psi(t)\|^2_{L^2(\Gamma)}.\end{aligned}$$

Then, estimate (3.23) gives

$$\begin{aligned}&\frac{d}{dt}(t\|\nabla\phi(t)\|^2_{L^2(\Omega)} + t\|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + t\lambda\|\psi(t)\|^2_{L^2(\Gamma)})\\ &\quad + t\|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + t\|\partial_t\psi(t)\|^2_{L^2(\Gamma)}\\ \leqslant&\|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + \lambda\|\psi(t)\|^2_{L^2(\Gamma)}\\ &\quad + Q(\|(u(0), v(0))\|^2_{\mathbb{W}})(\|\partial_t u(t)\|^2_{H^1(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)})\\ \leqslant&c(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)}) + Q(\|(u(0), v(0))\|^2_{\mathbb{W}})(\|\partial_t u(t)\|^2_{H^1(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)}).\end{aligned} \tag{3.24}$$

Integrating (3.24) with respect to t and using Theorem 3.2.5 and (3.19), we deduce

$$\begin{aligned}&t(\|\partial_t u(t)\|^2_{H^1(\Omega)} + \|\partial_t v(t)\|^2_{H^1(\Gamma)}) + \int_0^t (s\|\partial_{tt} u(s)\|^2_{H^{-1}(\Omega)} + s\|\partial_{tt} v(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ \leqslant&(c + Q(\|(u(0), v(0))\|^2_{\mathbb{W}})) \int_0^t (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)})\mathrm{d}s\\ \leqslant&Q(\|(u(0), v(0))\|^2_{\mathbb{W}}),\end{aligned} \tag{3.25}$$

$t \in [0, 1]$, for some monotonic Q which is independent of t. Having obtained estimate (3.25), we can rewrite problem (3.1) as a linear elliptic boundary value problem

$$\begin{cases} -\Delta u(t) = h_3(t) := w(t) - f(u(t)),\ u_{|\Gamma} = v\\ -\Delta_\Gamma v(t) + \lambda v(t) + \partial_n u(t) = h_4(t) := -\partial_t v(t) - g(v(t)).\end{cases} \tag{3.26}$$

According to Theorem 3.2.5, estimates (3.19) and (3.25), we have

$$\|h_3(t)\|^2_{H^1(\Omega)} + \|h_4(t)\|^2_{H^1(\Gamma)} \leqslant \frac{1}{t}Q(\|(u(0), v(0))\|^2_{\mathbb{W}}),\ t \in (0, 1],$$

for some monotonic function Q. Applying the $H^1 - H^3$ regularity theorem to the linear elliptic problem (3.26), we obtain

$$\begin{aligned}\|u(t)\|^2_{H^3(\Omega)} + \|v(t)\|^2_{H^3(\Gamma)} &\leqslant c(\|h_3(t)\|^2_{H^1(\Omega)} + \|h_4(t)\|^2_{H^1(\Gamma)})\\ &\leqslant \frac{1}{t}Q(\|(u(0), v(0))\|^2_{\mathbb{W}}),\ t \in (0, 1].\end{aligned}$$

□

We will also verify the uniqueness of the solution of (3.1) and the Lipschitz continuity with respect to the initial data.

Proposition 3.2.7. *Let the assumptions of Proposition 3.2.2 hold and let the functions* $(u_1(t), v_1(t))$ *and* $(u_2(t), v_2(t))$ *be two solutions of* (3.1)*. Then, we have*

$$\begin{aligned}\|u_1(t)-u_2(t)\|^2_{H^{-1}(\Omega)}+\|v_1(t)-v_2(t)\|^2_{L^2(\Gamma)}&\\ +\int_t^{t+1}(\|u_1(s)-u_2(s)\|^2_{H^1(\Omega)}+\|v_1(s)-v_2(s)\|^2_{H^1(\Gamma)})\mathrm{d}s&\\ \leqslant ce^{ct}\left(\|u_1(0)-u_2(0)\|^2_{H^{-1}(\Omega)}+\|v_1(0)-v_2(0)\|^2_{L^2(\Gamma)}\right),&\end{aligned} \tag{3.27}$$

where the positive constant c *is independent of the initial data.*

Proof:
Set $(u, w, v) = (u_1 - u_2, w_1 - w_2, v_1 - v_2)$. These functions satisfy the following problem:

$$\begin{cases} \partial_t u = \Delta w - w = -(-\Delta + I)w = -Aw,\ \partial_n w_{|\Gamma} = 0,\\ w = -\Delta u + (f(u_1) - f(u_2)) = -\Delta u + l_1(t)u,\\ \partial_t v = \Delta_\Gamma v - (g(u_1) - g(v_2)) - \lambda v - \partial_n u = \Delta_\Gamma v - \lambda v - \partial_n u - l_2(t)v,\ x \in \Gamma,\\ u_{|t=0} = u_1(0) - u_2(0),\ v_{|t=0} = v_1(0) - v_2(0),\\ u_{|\Gamma} = v \end{cases} \tag{3.28}$$

where $l_1(t) := \displaystyle\int_0^1 f'(su_1(t)+(1-s)u_2(t))\mathrm{d}s$ and $l_2(t) := \displaystyle\int_0^1 g'(sv_1(t)+(1-s)v_2(t))\mathrm{d}s$. We multiply the first equation of (3.28) scalarly by $A^{-1}u$, the second by u and the third by v and take the sum of the equations that we obtain. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2_{H^{-1}(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}) + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)} + \lambda\|v\|^2_{L^2(\Gamma)}&\\ = -\left(l_1(t)u(t), u(t)\right)_\Omega - \left(l_2(t)v(t), v(t)\right)_\Gamma.&\end{aligned} \tag{3.29}$$

Due to assumptions (3.2), $f'(v) \geqslant -c$ and $g'(v) \geqslant -c$, and thus estimate (3.29) implies

$$\begin{aligned}\frac{d}{dt}(\|u(t)\|^2_{H^{-1}(\Omega)}+\|v(t)\|^2_{L^2(\Gamma)}) + 2\|\nabla u(t)\|^2_{L^2(\Omega)} + 2\|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)} + 2\lambda\|v\|^2_{L^2(\Gamma)}&\\ \leqslant c(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}).&\end{aligned} \tag{3.30}$$

Using now the interpolation inequality $\|\mu\|^2_{L^2(\Omega)} \leqslant c\|\mu\|_{H^1(\Omega)}\|\mu\|_{H^{-1}(\Omega)}$ in order to estimate the first term in the right-hand side of (3.30), we obtain

$$\begin{aligned}\frac{d}{dt}(\|u(t)\|^2_{H^{-1}(\Omega)}+\|v(t)\|^2_{L^2(\Gamma)}) + 2\|u(t)\|^2_{H^1(\Omega)} + 2\|v(t)\|^2_{H^1(\Gamma)}&\\ \leqslant c(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)})&\\ \leqslant \|u(t)\|^2_{H^1(\Omega)} + c(\|u(t)\|^2_{H^{-1}(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}).&\end{aligned} \tag{3.31}$$

Applying Gronwall's inequality, we find

$$\begin{aligned}\|u(t)\|^2_{H^{-1}(\Omega)}+\|v(t)\|^2_{L^2(\Gamma)} + \int_0^t (\|u(s)\|^2_{H^1(\Omega)} + \|v(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ \leqslant (\|u(0)\|^2_{H^{-1}(\Omega)} + \|v(0)\|^2_{L^2(\Gamma)})e^{ct}.\end{aligned} \tag{3.32}$$

Integrating (3.31) with respect to $t \in [t, t+1]$ and using (3.32), we obtain

$$\begin{aligned}&\|u(t+1)\|^2_{H^{-1}(\Omega)}+\|v(t+1)\|^2_{L^2(\Gamma)} + \int_t^{t+1} (\|u(s)\|^2_{H^1(\Omega)} + \|v(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ &\qquad \leqslant c\int_t^{t+1} (\|u(s)\|^2_{H^{-1}(\Omega)} + \|v(s)\|^2_{L^2(\Gamma)})\mathrm{d}s + \|u(t)\|^2_{H^{-1}(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)} \\ &\qquad \leqslant ce^{ct}(\|u(0)\|^2_{H^{-1}(\Omega)} + \|v(0)\|^2_{L^2(\Gamma)}).\end{aligned} \tag{3.33}$$

Finally, estimates (3.32) and (3.33) give (3.27). □

Proposition 3.2.8. *Let the assumptions of Theorem 3.2.2 hold. Then, we have:*

$$\begin{aligned}&\|u_1(t) - u_2(t)\|^2_{H^1(\Omega)}+\|v_1(t) - v_2(t)\|^2_{H^1(\Gamma)} \\ &\qquad + \int_t^{t+1} (\|\partial_t u_1(s) - \partial_t u_2(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t v_1(s) - \partial_t v_2(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ &\qquad \leqslant ce^{C_0 t}\left(\|u_1(0) - u_2(0)\|^2_{H^1(\Omega)} + \|v_1(0) - v_2(0)\|^2_{H^1(\Gamma)}\right),\end{aligned} \tag{3.34}$$

where the positive constant C_0 depends on the initial data.

Proof:
Let $(u, w, v) = (u_1 - u_2, w_1 - w_2, v_1 - v_2)$. Then, these functions satisfy (3.28). Thus, multiplying scalarly the first, second and third equations of (3.28) by respectively $A^{-1}\partial_t u(t)$, $\partial_t u(t)$ and $\partial_t v(t)$, then integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|^2_{L^2(\Omega)}+\|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)} + \lambda\|v(t)\|^2_{L^2(\Gamma)}) + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)} \\ = -\left(l_1(t)u(t), \partial_t u(t)\right)_\Omega - \left(l_2(t)v(t), \partial_t v(t)\right)_\Gamma.\end{aligned} \tag{3.35}$$

It follows from Theorem 3.2.5, the assumptions (3.2) on the nonlinearities f and g and the embeddings $H^2(\Omega) \subset L^\infty(\Omega)$, $H^2(\Gamma) \subset L^\infty(\Gamma)$ that

$$\begin{aligned}\|\partial_t l_1(t)\|_{L^2(\Omega)} + \|\partial_t l_2(t)\|_{L^2(\Gamma)}+\|l_1(t)\|_{H^2(\Omega)} + \|l_2(t)\|_{H^2(\Gamma)} \\ \leqslant Q\left(\|(u_1(t), v_1(t))\|_{\mathbb{W}} + \|(u_2(t), v_2(t))\|_{\mathbb{W}}\right) \\ \leqslant C_0 := Q\left(\|(u_1(0), v_1(0))\|_{\mathbb{W}} + \|(u_2(0), v_2(0))\|_{\mathbb{W}}\right)\end{aligned} \tag{3.36}$$

consequently

$$\begin{aligned}&\left(l_1(t)u(t), \partial_t u(t)\right)_\Omega + \left(l_2(t)v(t), \partial_t v(t)\right)_\Gamma \\ &\leqslant C_0\|u(t)\|_{H^1(\Omega)}\|\partial_t u(t)\|_{H^{-1}(\Omega)} + C_0\|v(t)\|_{L^2(\Gamma)}\|\partial_t v(t)\|_{L^2(\Gamma)} \\ &\leqslant \frac{C_0^2}{2}(\|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}) + \frac{1}{2}(\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)}).\end{aligned}$$

Inserting the above estimate into the right-hand of (3.35), we obtain

$$\frac{d}{dt}(\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma v(t)\|^2_{L^2(\Gamma)} + \lambda\|v(t)\|^2_{L^2(\Gamma)}) + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)} \leqslant C_0^2 \left(\|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{H^1(\Gamma)}\right). \tag{3.37}$$

Applying Gronwall's inequality to relation (3.37), we obtain our desired estimate. □

3.3 Existence of solutions.

In this section, we establish the existence of a solution for problem (3.1). The proof is based on the priori estimates derived in the first section and a Faedo-Galerkin approximation scheme. To do this, let us consider the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{V} \to \mathbb{V}'$ defined by

$$\left\langle \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right\rangle_{\mathbb{V}',\mathbb{V}} = (\nabla u, \nabla u')_\Omega + \lambda(v, v')_\Gamma + (\nabla_\Gamma v, \nabla_\Gamma v')_\Gamma,$$

for all $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in \mathbb{V}$. We have that $\mathcal{A}$ is linear, selfadjoint, continuous, positive and invertible operator with compact inverse on $\mathbb{V}$ and that

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Delta u \\ -\Delta_\Gamma v + \lambda v + \partial_n u \end{pmatrix}, \ \forall \begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A}).$$

Consequently, let $\Phi_1 = \begin{pmatrix} \varphi_m \\ \chi_m \end{pmatrix}, \cdots, \Phi_m = \begin{pmatrix} \varphi_m \\ \chi_m \end{pmatrix}$ be an orthonormal family in $\mathbb{H}$ associated with the eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m$ of $\mathcal{A}$. We consider the approximate problem:

Find $U_m = \begin{pmatrix} u_m \\ v_m \end{pmatrix} = \sum_{i=1}^m c_i(t)\Phi_i = \begin{pmatrix} \sum_{i=1}^m c_i(t)\varphi_i \\ \sum_{i=1}^m c_i(t)\chi_i \end{pmatrix}$ such that

$$\begin{cases} \frac{d}{dt}(A^{-1}u_m, \varphi_j)_\Omega = -(\nabla u_m, \nabla\varphi_j)_\Omega + (\partial_n u, \chi_j)_\Gamma - (f(u_m), \varphi_j)_\Omega, \\ \frac{d}{dt}(v_m, \chi_j)_\Gamma = -(\nabla_\Gamma v, \nabla_\Gamma\chi_j)_\Gamma - \lambda(v, \chi_j)_\Gamma - (g(v_m), \chi_j)_\Gamma - (\partial_n u, \chi_j)_\Gamma, \end{cases} \tag{3.38}$$

for all $j = 1, \cdots, m$, that is to say, setting

$$\mathcal{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u) \\ g(v) \end{pmatrix} \text{ and } \mathcal{N}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A^{-1}u \\ v \end{pmatrix},$$

we search for U_m such that:

$$\frac{d}{dt}\left(\mathcal{N}U_m, \Phi_j\right)_{\mathbb{H}} + \left(\mathcal{A}U_m, \Phi_j\right)_{\mathbb{V}} + \left(\mathcal{F}(U_m), \Phi_j\right)_{\mathbb{H}} = 0, \ \forall j = 1, \cdots, m. \tag{3.39}$$

We define the following matrices

$$\mathbb{G}_0 = \left(\left(\mathcal{N} \begin{pmatrix} \varphi_i \\ \chi_i \end{pmatrix}, \begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix} \right)_{\mathbb{H}} \right)_{1 \leqslant i,j \leqslant m}$$

$$\mathbb{G}_1 = \left(\left(\lambda_i \begin{pmatrix} \varphi_i \\ \chi_i \end{pmatrix}, \begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix} \right)_{\mathbb{H}} \right)_{1 \leqslant i,j \leqslant m}$$

$$\mathbb{F} = \left(\left(\begin{pmatrix} f(\sum_{i=1}^{m} c_i(t)\varphi_i) \\ g(\sum_{i=1}^{m} c_i(t)\chi_i) \end{pmatrix}, \begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix} \right)_{\mathbb{H}} \right)_{1 \leqslant i,j \leqslant m}$$

and the column vector

$$c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_m(t) \end{pmatrix}.$$

Consequently, (3.39) can be rewriten in the following form

$$\mathbb{G}_0 c'(t) + \mathbb{G}_1 c(t) + \mathbb{F}(c(t)) = 0. \tag{3.40}$$

We have that $\mathbb{G}_0$ is symmetric and positive-definite, hence non-singular. $\mathbb{G}_0$ and $\mathbb{G}_1$ are uniformly continuous with respect to t. Applying Cauchy's theorem for ODE's, we find that there exists a small time $t_m \in (0, T)$ such that the equation $c'(t) = -\mathbb{G}_0^{-1}\mathbb{G}_1 c(t) - \mathbb{G}_0^{-1}\mathbb{F}(c(t))$ holds for all $t \in [0, t_m]$. Now, based on the uniform a priori estimates with respect to t, derived for the solution (u, v) of (3.1), we obtain, in particular, that any local solution of (3.40) is actually a global solution that is defined on the whole interval $[0, T]$. It remains then to pass to the limit as $m \to \infty$.

According to the a priori estimates derived in the Section 2, we have

$$\|u_m\|_{L^\infty([0,T];H^2(\Omega))} + \|v_m\|_{L^\infty([0,T];H^2(\Gamma))} \leqslant c$$

$$\|\partial_t u_m\|_{L^2([0,T];H^1(\Omega))} + \|\partial_t v_m\|_{L^2([0,T];H^1(\Gamma))} \leqslant c.$$

The approximate chemical potential $w_m = -\Delta u_m + f(u_m),\ n \in \mathbb{N}$, satisfies

$$\|w_m\|_{L^\infty([0,T];H^1(\Omega))} \leqslant c,$$

where c depends on $\Omega,\ \Gamma,\ u_0,\ v_0$ and T but is independent of n and t. From this point on, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not relabeled. Thus, we observe that weak and weak star compactness results applied to the sequences $U_m = (u_m, v_m)$ entail that there exists the function $U = (u, v)$ such that, as $m \to \infty$, the following properties hold

$$U_m \to U \text{ weakly star in } L^\infty([0,T]; H^2(\Omega) \times H^2(\Gamma)), \tag{3.41}$$

$$\partial_t U_m \to \partial_t U \text{ weakly in } L^2([0,T]; H^1(\Omega) \times H^1(\Gamma)), \tag{3.42}$$

$w_m \to \zeta$ weakly star in $L^\infty([0,T];H^1(\Omega))$(we prove later that $\zeta = -\Delta u + f(u) = w$).

It follows from (3.41), (3.42) and the compactness theorem of J.P Aubin-J.L Lions that

$$U_m \to U \text{ strongly in } C([0,T];H^{2-\delta}(\Omega)\times H^{2-\delta}(\Gamma)) \tag{3.43}$$

with $\delta > 0$ sufficiently small. Moreover, due to the embedding $H^{2-\delta}(\Omega) \subset C(\Omega)$, for $\delta \in (0,1/2)$, since $\Omega \subset \mathbb{R}^n$ with $n \leqslant 3$, the convergence (3.43) yields

$$U_m \to U \text{ strongly in } C([0,T];C(\Omega)\times C(\Gamma)).$$

We have

$$\|f(u_m) - f(u)\|^2_{L^2(\Omega)} \leqslant c\|u_m - u\|^2_{L^2(\Omega)}$$

$$\|g(v_m) - g(v)\|^2_{L^2(\Gamma)} \leqslant c\|v_m - v\|^2_{L^2(\Gamma)}.$$

Consequently, it follows that

$$f(u_m) \to f(u) \text{ strongly in } C([0,T];L^2(\Omega)) \tag{3.44}$$

and

$$g(v_m) \to g(v) \text{ strongly in } C([0,T];L^2(\Gamma)). \tag{3.45}$$

Thus, passing to the limit in (3.39) and using the above convergence properties, we immediately have that (u,v) satisfies (3.1). We also deduce from (3.44) that $\zeta = w$ and $w_m \to w$ weakly star in $L^\infty([0,T];H^1(\Omega))$.

Corollary 3.3.1. *Let $(u_0,v_0) \in \mathbb{W}$ and suppose that the nonlinearities f and g satisfy assumptions* (3.2)*. Then, problem* (3.1) *has a unique solution $(u(t),v(t))$ that belongs to the space $C([0,T];\mathbb{W})$. Moreover this problem defines a semigroup $S(t)$ in the phase space $\mathbb{W}$ by*

$$S(t) : \mathbb{W} \to \mathbb{W}, \tag{3.46}$$

such that

$$S(t)(u_0,v_0) = (u(t),v(t)). \tag{3.47}$$

Proof:
The uniqueness of a solution has been proved in Proposition 3.2.7 and the existence of solutions in the space $\mathbb{W}$ has been proved above using a Faedo-Galerkin approximation scheme. Thus, problem (3.1) generates a semigroup $S(t)$ on $\mathbb{W}$ given by (3.46), where $(u(t),v(t))$ solves (3.1) with initial data in $\mathbb{W}$. □

3.4 Global and exponential attractor

We introduce the following ball $\mathbb{B}$ with sufficiently large radius R in the space $H^3(\Omega)\times H^3(\Gamma)$:

$$\mathbb{B}_R := \left\{(u,v) \in H^3(\Omega)\times H^3(\Gamma), \|(u,v)\|_{H^3(\Omega)\times H^3(\Gamma)} \leqslant R\right\}.$$

Then, obviously, $\mathbb{B}_R \subset \mathbb{W}$ and

$$\|\mathbb{B}_R\|_{\mathbb{W}} \leqslant c_R, \tag{3.48}$$

where the constant c_R depends on R. Moreover, due to the dissipative estimate (3.18) and the smoothing property (3.22), there exists a sufficiently large $\overline{R}$ and a time t_* which are independent of t such that $\mathbb{B}_{\overline{R}} := \mathbb{B}$ is a bounded absorbing set for the semigroup $S(t)$ acting on $\mathbb{W}$ and

$$S(t)\mathbb{B} \subset \mathbb{B},\ t \geqslant t_*.$$

We deduce from above the existence of a connected compact global attractor $\mathcal{A} \subset \mathbb{W} \cap (H^3(\Omega) \times H^3(\Gamma))$ for the semigroup $S(t)$ associated to the system (3.1).

Lemma 3.4.1. *For, every $(u_1(0), v_1(0))$ and $(u_2(0), v_2(0))$ belonging to $\mathbb{B}$, the corresponding solutions of problem (3.1) satisfy the following estimate*

$$\begin{aligned}\|u_1(t) - u_2(t)\|^2_{H^1(\Omega)} + &\|v_1(t) - v_2(t)\|^2_{H^1(\Gamma)} \\ &\leqslant c_{\overline{R}} e^{ct} \frac{t+1}{t} \|(u_1(0) - u_2(0), v_1(0) - v_2(0))\|^2_{\mathbb{L}},\ t > 0,\end{aligned} \tag{3.49}$$

where the positive constant $c_{\overline{R}}$ depends only on $\overline{R}$ and is independent of t.

Proof:
Let $(u, w, v) := (u_1 - u_2, w_1 - w_2, v_1 - v_2)$. Then, these functions satisfy equations (3.28). From Proposition 3.2.7, we have

$$\|(u(t), v(t))\|^2_{\mathbb{L}} + \int_0^t (\|u(s)\|^2_{H^1(\Omega)} + \|v(s)\|^2_{H^1(\Gamma)}) \mathrm{d}s \leqslant c e^{ct} \|(u(0), v(0))\|^2_{\mathbb{L}}. \tag{3.50}$$

To verify estimate (3.49), we multiply scalarly the first equation of (3.28) scalarly by $tA^{-1}\partial_t u$, the second by $t\partial_t u$ and the third by $t\partial_t v$. Arguing as in the derivation of (3.23), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(t\|\nabla u(t)\|^2_{L^2(\Omega)} + t\|v(t)\|^2_{L^2(\Gamma)} + \lambda t\|v(t)\|^2_{L^2(\Gamma)}) + t\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + t\|v(t)\|^2_{L^2(\Gamma)} \\ =&\frac{1}{2}\left(\|\nabla u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)} + \lambda\|v(t)\|^2_{L^2(\Gamma)}\right) - (f(u_1(t)) - f(u_2(t)), t\partial_t u(t))_\Omega \\ &- (g(v_1(t)) - g(v_2(t)), t\partial_t v(t))_\Gamma.\end{aligned}$$

Due to estimates (3.18) and (3.48) and to the embedding $H^2(\Omega) \subset C(\Omega)$, we have

$$\|u_i(t)\|_{L^\infty(\Omega)} + \|v_i(t)\|_{L^\infty(\Gamma)} \leqslant c(\|u_i(t)\|_{H^2(\Omega)} + \|v_i(t)\|_{H^2(\Gamma)}) \leqslant c_{\overline{R}},\ \ i = 1, 2,\ t \geqslant 0. \tag{3.51}$$

Estimating the last two terms in the right-hand side of the equation above as follows

$$\begin{aligned}&(f(u_1(t)) - f(u_2(t)), t\partial_t u(t))_\Omega + (g(v_1(t)) - g(v_2(t)), t\partial_t v(t))_\Gamma \\ =&(l_1(t)u(t), t\partial_t u(t))_\Omega + (l_2(t)v(t), t\partial_t v(t))_\Gamma \\ \leqslant&t\|l_1(t)\|_{L^\infty(\Omega)}\|u(t)\|_{H^1(\Omega)}\|\partial_t u(t)\|_{H^{-1}(\Omega)} + t\|l_2(t)\|_{L^\infty(\Gamma)}\|v(t)\|_{L^2(\Gamma)}\|\partial_t v(t)\|_{L^2(\Gamma)} \\ \leqslant&1/2\left(t\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + t\|\partial_t v(t)\|^2_{L^2(\Gamma)}\right) + c_{\overline{R}} t\left(\|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}\right),\end{aligned} \tag{3.52}$$

where we have used estimate (3.51). We obtain

$$\begin{aligned}
&\frac{d}{dt}(t\|\nabla u(t)\|^2_{L^2(\Omega)} + t\|v(t)\|^2_{L^2(\Gamma)} + \lambda t\|v(t)\|^2_{L^2(\Gamma)}) + t\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + t\|v(t)\|^2_{L^2(\Gamma)} \\
\leqslant &\left(\|\nabla u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)} + \lambda\|v(t)\|^2_{L^2(\Gamma)}\right) + c_{\overline{R}} t\left(\|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}\right) \\
\leqslant &c_{\overline{R}}(t+1)\left(\|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{H^1(\Gamma)}\right).
\end{aligned} \tag{3.53}$$

Integrating with respect to t and using (3.50), we infer

$$\begin{aligned}
\|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{H^1(\Gamma)} &\leqslant c_{\overline{R}} e^{ct}\frac{t+1}{t}\|(u(0),v(0))\|^2_{\mathbb{L}} \\
&\leqslant c_{\overline{R}} e^{ct}\frac{t+1}{t}\|(u(0),v(0))\|^2_{L^2(\Omega)\times L^2(\Gamma)},
\end{aligned} \tag{3.54}$$

and the lemma is proven. □

Lemma 3.4.2. *Let the assumptions of Lemma 3.4.1 hold. Then, for every $t > 0$, we have the following smoothing estimate*

$$\begin{aligned}
&\|u_1(t) - u_2(t)\|^2_{H^2(\Omega)} + \|v_1(t) - v_2(t)\|^2_{H^2(\Gamma)} \\
&\qquad \leqslant c_{\overline{R}} e^{ct}\frac{t+1}{t}(\|(u_1(0) - u_2(0), v_1(0) - v_2(0))\|^2_{\mathbb{L}}),
\end{aligned} \tag{3.55}$$

where the positive constant $c_{\overline{R}}$ depends only on $\overline{R}$ and independent of t.

Proof:
Recall that the function u and v satisfy (3.28). Differentiating now all equations of (3.28) with respect of t, multiplying scalarly the first, second and third equations of (3.28) by $A^{-1}t\partial_t u$, $t\partial_t u$ and $t\partial_t v$, respectively, and summing up the relations obtained, we have

$$\begin{aligned}
&t\frac{d}{dt}(\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)}) \\
&\quad + 2t\|\nabla\partial_t u\|^2_{L^2(\Omega)} + 2t\|\nabla_\Gamma \partial_t v(t)\|^2_{L^2(\Gamma)} + 2t\lambda\|\partial_t v(t)\|^2_{L^2(\Gamma)} \\
= &-2t\left(l_1(t)\partial_t u(t), \partial_t u(t)\right)_\Omega - 2t\left(\partial_t l_1(t) u(t), \partial_t u(t)\right)_\Omega - 2t\left(l_2(t)\partial_t v(t), \partial_t v(t)\right)_\Gamma \\
&\quad - 2t\left(\partial_t l_2(t) v(t), \partial_t v(t)\right)_\Gamma.
\end{aligned}$$

Using (3.36) and the embeddings $H^1(\Omega) \subset L^4(\Omega)$, $H^1(\Gamma) \subset L^4(\Gamma)$, we have

$$\begin{aligned}
&\left(l_1(t)\partial_t u(t), \partial_t u(t)\right)_\Omega + \left(\partial_t l_1(t) u(t), \partial_t u(t)\right)_\Omega + \left(l_2(t)\partial_t v(t), \partial_t v(t)\right)_\Gamma \\
&\quad + \left(\partial_t l_2(t) v(t), \partial_t v(t)\right)_\Gamma \\
\leqslant &\|l_1(t)\|_{L^\infty(\Omega)}\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|l_2(t)\|_{L^\infty(\Gamma)}\|\partial_t v(t)\|^2_{L^2(\Gamma)} \\
&\quad + \|\partial_t l_1(t)\|_{L^2(\Omega)}\|u(t)\|_{L^4(\Omega)}\|\partial_t u(t)\|_{L^4(\Omega)} + \|\partial_t l_2(t)\|_{L^2(\Gamma)}\|v(t)\|_{L^4(\Gamma)}\|\partial_t v(t)\|_{L^4(\Gamma)} \\
\leqslant &c\|l_1(t)\|_{L^\infty(\Omega)}\|\partial_t u(t)\|_{H^{-1}(\Omega)}\|\partial_t u(t)\|_{H^1(\Omega)} + \|l_2(t)\|_{L^\infty(\Gamma)}\|\partial_t v(t)\|^2_{L^2(\Gamma)} \\
&\quad + \|\partial_t l_1(t)\|_{L^2(\Omega)}\|u(t)\|_{L^4(\Omega)}\|\partial_t u(t)\|_{L^4(\Omega)} + \|\partial_t l_2(t)\|_{L^2(\Gamma)}\|v(t)\|_{L^4(\Gamma)}\|\partial_t v(t)\|_{L^4(\Gamma)} \\
\leqslant &\frac{1}{2}c_{\overline{R}}(\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)} + \|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{H^1(\Gamma)}) \\
&\quad + \frac{1}{2}(\|\partial_t u(t)\|^2_{H^1(\Omega)} + \|\partial_t v(t)\|^2_{H^1(\Gamma)}),
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{d}{dt}[t(\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)})] + t(\|\partial_t u(t)\|^2_{H^1(\Omega)} + \|\partial_t v(t)\|^2_{H^1(\Gamma)}) \\
\leqslant & c_{\overline{R}} t(\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)} + \|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{H^1(\Gamma)}) \\
&+ \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)} \\
\leqslant & (c_{\overline{R}} t + 1)(\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)}) + c_{\overline{R}} t(\|u(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{H^1(\Gamma)})
\end{aligned} \tag{3.56}$$

where $c_{\overline{R}}$ depends on the norm of the initial data in $\mathbb{B}$ but independent of t. Integrating (3.56) over $(0,t)$ and using Proposition 3.2.8 and estimate (3.50), we obtain that

$$t(\|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Gamma)}) \leqslant c_{\overline{R}} e^{ct}(t+1)(\|u(0)\|^2_{H^{-1}(\Omega)} + \|v(0)\|^2_{L^2(\Gamma)}). \tag{3.57}$$

The estimate of Lemma 3.4.2 is then a consequence of (3.57), standard parabolic arguments and maximum principle as in the proof of Theorem 3.2.5.

□

Lemma 3.4.3. *Let (u,v) be the solution of (3.1) with initial data $(u(0),v(0))$ in $\mathbb{B}$. Then, (u,v) is Hölder continuous with respect to t, that is, for every $t \geqslant t_* > 0$ and $0 \leqslant s \leqslant 1$, we have*

$$\|u(t+s) - u(t)\|_{H^{-1}(\Omega)} + \|v(t+s) - v(t)\|_{L^2(\Gamma)} \leqslant Q(\|(u(0),v(0))\|^2_{\mathbb{W}}) s^{1/2}. \tag{3.58}$$

Proof:
From estimate (3.11), we have that for every $t \in [t_*, T]$, with T fixed, the following

$$\int_t^{t+1} (\|\partial_t u(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t v(s)\|^2_{L^2(\Gamma)}) \mathrm{d}s \leqslant Q(\|(u(0),v(0))\|^2_{\mathbb{W}}) \leqslant c_{\overline{R}},$$

for some positive constant $c_{\overline{R}}$ which depends on $\overline{R}$. Moreover, for every $t \geqslant t_* > 0$ and $0 \leqslant s \leqslant 1$, we have

$$\begin{aligned}
\|u(t+s) - u(t)\|_{H^{-1}(\Omega)} + & \|v(t+s) - v(t)\|_{L^2(\Gamma)} \\
&= \| \int_t^{t+s} \partial_t u(\sigma) \mathrm{d}\sigma \|_{H^{-1}(\Omega)} + \| \int_t^{t+s} \partial_t v(\sigma) \mathrm{d}\sigma \|_{L^2(\Gamma)} \\
&\leqslant \int_t^{t+s} (\|\partial_t u(\sigma)\|_{H^{-1}(\Omega)} + \|\partial_t v(\sigma)\|_{L^2(\Gamma)}) \mathrm{d}\sigma \\
&\leqslant c_{\overline{R}} s^{1/2}.
\end{aligned} \tag{3.59}$$

Thus, the Lemma is proved. □

In order to prove the existence of an exponential attractor, we will apply the following theorem (see [10]),

Theorem 3.4.4. *Let E and E_1 be two Banach spaces such that E_1 is compactly embedded into E. Let X be a bounded subset of E_1 and consider a nonlinear map $S : X \to X$ satisfying the smoothing property*

$$\|Sx_1 - Sx_2\|_{E_1} \leqslant C\|x_1 - x_2\|_E, \tag{3.60}$$

for all $x_1,\ x_2 \in X$, *where* $C > 0$ *depends on* X. *Then the discrete dynamical system* $(X, (S^*)^n)$ *possesses a discrete exponential attractor* $\mathcal{M}^d \subset E$, *that is, a compact set with finite fractal dimension such that*

$$S(\mathcal{M}^d) \subset \mathcal{M}^d, \tag{3.61}$$

$$dist_E(S^n(X), \mathcal{M}^d) \leqslant C_X e^{-\rho n}, \quad n \in \mathbb{N}, \tag{3.62}$$

where C_X *and* ρ *are positive constants independent of* n.

We will apply this theorem with $X = \mathbb{B}$, $E = \mathbb{L}$, $E_1 = H^2(\Omega) \times H^2(\Gamma)$ and $S = S(t_*)$. Thus, the mapping $S : X \to X$ satisfies the smoothing property (3.60). Therefore, Theorem 3.4.4 applies to S and there exists a compact set $\mathcal{M}^d \subset \mathbb{B}$ of finite fractal dimension (with respect to the metric in $\mathbb{L}$) that satisfies (3.61) and (3.62). In order to extend this result to continuous times, we use the standard formula

$$\mathcal{M}^c = \cup_{t\in[t_*,2t_*]} S(t)\mathcal{M}^d.$$

Then, since the semigroup $S(t)$ is uniformly Lipschitz continuous on $[t_*, 2t_*] \times \mathbb{B}$ in the norm of $\mathbb{L}$ (Lemma 3.4.3), we have that $\mathcal{M}^c$ is a compact set with finite fractal dimension such that

$$\dim_F(\mathcal{M}^c, \mathbb{L}) \leqslant \dim_F(\mathcal{M}^d, \mathbb{L}) + 2. \tag{3.63}$$

Chapitre 4

Numerical Analysis of a Cahn-Hilliard Type Equation With Dynamic Boundary Conditions

Analyse numérique d'une équation de type Cahn-Hilliard avec des conditions dynamiques sur le bord

4.1 Introduction

We consider the following problem in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$:

$$
\begin{aligned}
u_t &= \Delta w - w, \qquad t > 0,\ x \in \Omega, && (4.1)\\
w &= f(u) - \Delta u, \qquad t > 0,\ x \in \Omega, && (4.2)\\
u_t &= \Delta_\Gamma u - \lambda u - g(u) - \partial_n u, \qquad t > 0,\ x \in \Gamma, && (4.3)\\
\partial_n w &= 0, \qquad t > 0,\ x \in \Gamma, && (4.4)
\end{aligned}
$$

where Δ_Γ is the Laplace-Beltrami operator on the boundary Γ, f and g are given nonlinear interaction functions and λ is some given positive constant. The boundary condition (4.3) will be interpreted as an additional second-order parabolic equation on the boundary Γ.

Problem (4.1)-(4.2) was introduced by Karali and Katsoulakis in [30] as a simplification of a mesoscopic model for multiple microscopic mechanisms in surface processes such that surface diffusion and adsorption-desorption and studied in [28], [31] and [32]; questions related to the well-posedness and to the asymptotic behavior, such as the existence of the global attractor and an exponential attractor have been

answered under various assumptions on the nonlinearities and when the system is endowed with Dirichlet and Neumann boundary conditions; Here u is the order parameter and corresponds to a rescaled density of atoms and w is the chemical potential. Now, the question of how the phase separation process is influenced by the presence of walls has recently gained much attention and was mainly studied for polymer mixtures when the boundary conditions are given by (4.3)-(4.4). The well-posedness and the long time behavior of problem (4.1)-(4.4) were studied in [28] and [27], respectively with regular and singular potentials.
The chapter is organized as follows. We first introduce in Section 2 some notation and assumptions and, in Section 3, we study a space discretization of (4.1)-(4.4) by a Galerkin method. In Section 4, we prove optimal error estimates for the difference between the approximate and the exact solution $u^h - u$ in energy norms and weaker norms as the mesh step h tends to 0, where u^h is the solution of the space semi-discrete scheme and u is the solution of the continuous problem. In Section 5, we study the numerical stability of the fully discrete problem obtained by applying the Euler implicit method to the space semi-discrete problem. In particular, we show that this fully discrete problem is unconditionally stable and the solution converges to equilibrium as $h \to 0$. Finally, numerical simulations in two space dimensions are presented in Section 6.

4.2 Assumptions and notation

In what follows, we consider Ω to be a $2d$ or $3d$ slab, i.e.

$$\Omega = \Pi_{i=1}^{d-1}(\mathbb{R}/(L_i\mathbb{Z})) \times (0, L_d),\ L_i > 0, i = 1, ..., d,\ d = 2 \text{ or } 3,$$

with smooth boundary

$$\Gamma = \partial\Omega = \Pi_{i=1}^{d-1}(\mathbb{R}/(L_i\mathbb{Z})) \times \{0, L_d\}\,.$$

More precisely, when $d = 2$, Ω is the rectangle $(0, L_1) \times (0, L_2)$ and u, w are periodic in the x_1-direction while the boundary conditions in problem (4.1) are valid for $x_2 = 0$ and $x_2 = L_2$; when $d = 3$, Ω is a parallelepiped $(0, L_1) \times (0, L_2) \times (0, L_3)$, u and w are periodic in the x_1 and x_2-directions and the boundary conditions in problem (4.1) are valid for $x_3 = 0$ and $x_3 = L_3$.
We assume that the nonlinearities f and g belong to $C^2(\mathbb{R}, \mathbb{R})$ and satisfy the following standard dissipativity assumptions

$$\liminf_{|v|\to\infty} f'(v) > 0,\ \liminf_{|v|\to\infty} g'(v) > 0. \tag{4.5}$$

Typical choices are

$$f(v) = v^3 - v \text{ and } g(v) = kv - h \quad (v \in \mathbb{R}), \tag{4.6}$$

where $k > 0$ and $h \in \mathbb{R}$ are constants. The evolution boundary value problem (4.1)-(4.4) is completed by the initial condition $u(0) = u_0$.
We introduce the space

$$V = \left\{ v \in H_p^1(\Omega) \text{ and } v(\cdot, 0),\ v(\cdot, L_d) \in H_{per}^1\left(\Pi_{i=1}^{d-1}]0, L_i[\right) \right\},$$

where H^1_{per} is the classical space of periodic functions and

$$H^1_p(\Omega) = \left\{ v \in H^1(\Omega),\ v \text{ is periodic in the } x_1, \cdots, x_{d-1} - \text{directions} \right\}.$$

Then, V is a Hilbert space for the Hilbertian norm

$$\|v\|_V = \left(\|v\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Gamma)} \right)^{1/2}.$$

The space V can also be seen as the closure for the $\|\cdot\|_V-$norm of $C^1(\overline{\Omega})$ with the periodicity condition; V is continuously and densely embedded in $H^1(\Omega)$ and is isometric to the closed subspace $\tilde{V}$ of $H^1_p(\Omega) \times H^1_{per}(\Gamma)$ defined by:

$$\tilde{V} = \left\{ (u, \varphi) \in H^1_p(\Omega) \times H^1_{per}(\Gamma),\ u_{|\Gamma} = \varphi \text{ in the sense of traces} \right\}.$$

We denote by $(\cdot,\cdot)_\Omega$ the $L^2(\Omega)-$scalar product and by $|\cdot|_0$ the $L^2(\Omega)-$norm. Similarly, $(\cdot,\cdot)_\Gamma$ denotes the $L^2(\Gamma)-$scalar product and $|\cdot|_{0,\Gamma}$ the associated norm. As a shortcut, we will denote by $\|\cdot\|_k$ the Hilbertian norm in $H^k(\Omega)$ and $|\cdot|_k$ the associated seminorm, i.e.

$$\forall v \in H^k(\Omega), \|v\|^2_k = \sum_{|\alpha| \leqslant k} |\partial^\alpha v|^2_0 \text{ and } |v|^2_k = \sum_{|\alpha| = k} |\partial^\alpha v|^2_0.$$

Similarly, $\|\cdot\|_{k,\Gamma}$ denotes the $H^k(\Gamma)-$norm and $|\cdot|_{k,\Gamma}$ the associated seminorm.

We note that from (4.5), we can deduce that (see [7])

$$F(v) \geqslant c_1 v^2 - c_2, \quad G(v) \geqslant c_1 v^2 - c_2, \quad \forall v \in \mathbb{R}, \tag{4.7}$$

for some constants $c_1 > 0$ and $c_2 \geqslant 0$ and where F is an antiderivative of f and G is an antiderivative of g.

4.3 The semi-discrete scheme

The variational formulation of (4.1)-(4.4) reads

$$\left\{ \begin{array}{ll} (u_t, \varphi)_\Omega & = -(\nabla w, \nabla \varphi)_\Omega - (w, \varphi)_\Omega, \\ (w, \chi)_\Omega & = (f(u), \chi)_\Omega + (\nabla u, \nabla \chi)_\Omega + (\nabla_\Gamma u, \nabla_\Gamma \chi)_\Gamma + \lambda (u, \chi)_\Gamma + (g(u), \chi)_\Gamma \\ & \quad + (u_t, \chi)_\Gamma, \end{array} \right. \tag{4.8}$$

for all $\varphi \in H^1_p(\Omega)$ and for all $\chi \in V$.
We introduce the functional $\mathcal{E} : V \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}x + \int_\Gamma \left(\frac{1}{2} |\nabla_\Gamma u|^2 + \frac{\lambda}{2} |u|^2 + G(u) \right) \mathrm{d}\sigma. \tag{4.9}$$

If u is a regular solution of (4.1)-(4.4), then u dissipates $\mathcal{E}$. Indeed, choosing $\varphi = w$ and $\chi = u_t$ in (4.8) and subtracting the two equations, we obtain

$$\frac{d}{dt} \mathcal{E}(u(t)) = -\int_\Omega |\nabla w|^2 \mathrm{d}x - \int_\Omega |w|^2 \mathrm{d}x - \int_\Gamma |u_t|^2 \mathrm{d}\sigma, \quad \forall t \geqslant 0. \tag{4.10}$$

For the space discretization, we consider a quasiuniform family of decompositions $\left\{\Omega^h\right\}_h$ of $\Pi_{i=1}^d[0, L_i]$ into $d-$simplices which take into account the periodic boundary conditions on Ω, so that $\left\{\Omega^h\right\}_h$ is also a triangulation of $\overline{\Omega}$. The triangulation Ω^h of $\overline{\Omega}$ induces a triangulation Γ^h of Γ into $d-1$ simplices in a natural way. For a given triangulation $\Omega^h = \cup_{T\in\Omega^h} T$, we define V^h as the usual P^1 conforming finite element space

$$V^h = \left\{v^h \in C^0(\overline{\Omega}), v^h|_T \text{ is affine } \forall T \in \Omega^h\right\}.$$

For $u \in C^0(\overline{\Omega})$, let $I^h u$ denote the P^1 interpolate of u on Ω^h, i.e. $I^h u$ is the unique function in V^h which takes the same values as u on the nodes of the triangulation. We have the following standard approximation results, where $C > 0$ denotes a constant which depends only on $\left\{\Omega^h\right\}_h$

$$\forall u \in H_p^2(\Omega),\ |u - I^h u|_0 + h|u - I^h u|_1 \leqslant Ch^2|u|_2 \tag{4.11}$$

and

$$\forall \varphi \in H_{per}^2(\Gamma),\ |\varphi - I^h\varphi|_{0,\Gamma} + h|\varphi - I^h\varphi|_{1,\Gamma} \leqslant Ch^2|\varphi|_{2,\Gamma}. \tag{4.12}$$

Moreover, we have the following inverse estimate (see [15])

$$\forall v^h \in V^h,\ \|v^h\|_{C^0(\overline{\Omega})} \leqslant Ch^{-d/2}|v^h|_0, \tag{4.13}$$

where d is the space dimension.

The space semi-discrete version of (4.8) reads:
Find $(u^h, w^h) : [0, T] \to V^h \times V^h$ such that

$$\begin{cases} (u_t^h, \varphi)_\Omega &= -(\nabla w^h, \nabla\varphi)_\Omega - (w^h, \varphi)_\Omega, \\ (w^h, \chi)_\Omega &= (f(u^h), \chi)_\Omega + (\nabla u^h, \nabla\chi)_\Omega + (\nabla_\Gamma u^h, \nabla_\Gamma\chi)_\Gamma + \lambda(u^h, \chi)_\Gamma + (g(u^h), \chi)_\Gamma \\ &\quad +(u_t^h, \chi)_\Gamma, \end{cases} \tag{4.14}$$

for all $\varphi,\ \chi \in V^h$.
We define the operator $G^h : L^2(\Omega) \to V^h$, $v \to G^h v$, where $G^h v$ is the unique solution of the problem

$$(\nabla G^h v, \nabla\chi)_\Omega + (G^h v, \chi)_\Omega = (v, \chi)_\Omega,\ \forall\chi \in V^h. \tag{4.15}$$

We also define the discrete norm

$$|v|_{-1,h} = (G^h v, v)_\Omega^{1/2} = \left(|\nabla G^h v|_0^2 + |G^h v|_0^2\right)^{\frac{1}{2}},\ \forall v \in L^2(\Omega).$$

The norm $|\cdot|_{-1,h}$ is a discrete version of the H^{-1}-norm. We note that G^h is selfadjoint and positive definite on $L^2(\Omega)$. Indeed, for $\chi = G^h v$ in (4.15), we have

$$(G^h v, v)_\Omega = |\nabla G^h v|_0^2 + |G^h v|_0^2 \geqslant 0$$

and

$$(v, G^h v')_\Omega = (\nabla G^h v, \nabla G^h v')_\Omega + (G^h v, G^h v')_\Omega = (v', G^h v)_\Omega,$$

for all $v, v' \in L^2(\Omega)$. Moreover, the following interpolation inequalities hold

$$|v^h|_0^2 \leqslant |v^h|_{-1,h}\|v^h\|_1,\ \forall v^h \in V^h, \tag{4.16}$$

and

$$|v|_{-1,h} \leqslant |v|_0, \ \forall v \in L^2(\Omega). \tag{4.17}$$

In order to prove (4.16), we write

$$\begin{aligned} |v^h|_0^2 &= (\nabla G^h v^h, \nabla v^h)_\Omega + (G^h v^h, v^h)_\Omega \\ &\leqslant |\nabla G^h v^h|_0 |\nabla v^h|_0 + |G^h v^h|_0 |v^h|_0 \\ &\leqslant (|\nabla G^h v^h|_0^2 + |G^h v^h|_0^2)^{\frac{1}{2}} (|\nabla v^h|_0^2 + |v^h|_0^2)^{\frac{1}{2}} \\ &= |v^h|_{-1,h} \|v^h\|_1. \end{aligned}$$

To prove (4.17), we write

$$|v|_{-1,h}^2 = (G^h v, v)_\Omega \leqslant |G^h v|_0 |v|_0$$

and

$$|G^h v|_0^2 \leqslant \|G^h v\|_1^2 = |\nabla G^h v|_0^2 + |G^h v|_0^2 = |v|_{-1,h}^2.$$

Proposition 4.3.1. *For every $u_0^h \in V^h$, problem (4.14) has a unique solution*

$$(u^h, w^h) \in C^1([0, +\infty); V^h \times V^h),$$

such that $u^h(0) = u_0^h$. Moreover,

$$\mathcal{E}(u^h(t)) + \int_0^t (|w^h|_1^2 + |w^h|_0^2 + |u_t^h|_{0,\Gamma}^2) \mathrm{d}s \leqslant \mathcal{E}(u^h(0)), \ \forall t \geqslant 0, \tag{4.18}$$

where $\mathcal{E}$ is defined by (4.9).

Proof:
Let $(\varphi_1, \ldots, \varphi_m)$ be an orthonormal basis of V^h for the $L^2(\Omega)$-scalar product. We seek for $u^h(t) = \sum_{i=1}^m u_i(t)\varphi_i$ and $w^h(t) = \sum_{i=1}^m w_i(t)\varphi_i$. We define the matrices

$$A_{ij} = (\nabla\varphi_i, \nabla\varphi_j)_\Omega, \ (M_\Gamma)_{ij} = (\varphi_i, \varphi_j)_\Gamma \text{ and } (A_\Gamma)_{ij} = (\nabla_\Gamma\varphi_i, \nabla_\Gamma\varphi_j)_\Gamma,$$

for $1 \leqslant i, j \leqslant m$, the vectors $U = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$, $W = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$, and the functions

$$F^h(U) = \begin{pmatrix} (f(u^h), \varphi_1)_\Omega \\ \vdots \\ (f(u^h), \varphi_m)_\Omega \end{pmatrix}, \ G_\Gamma^h(U) = \begin{pmatrix} (\lambda u^h + g(u^h), \varphi_1)_\Gamma \\ \vdots \\ (\lambda u^h + g(u^h), \varphi_m)_\Gamma \end{pmatrix}.$$

Then (4.14) can be written as

$$\begin{pmatrix} (A+I) & I \\ -I & M_\Gamma \end{pmatrix} \begin{pmatrix} W \\ U' \end{pmatrix} = - \begin{pmatrix} 0 \\ AU + F^h(U) + A_\Gamma U + G_\Gamma^h(U) \end{pmatrix}. \tag{4.19}$$

Let B denote the square matrix of size $2M$ in the left-hand side of (4.19). We claim that B is invertible. Indeed, let $X, Y \in \mathbb{R}^M$. We have

$$\begin{aligned}(X^t, Y^t)B\begin{pmatrix} X \\ Y \end{pmatrix} &= X^t(A+I)X + X^tY - Y^tX + Y^tM_\Gamma Y \\ &= (\nabla x^h, \nabla x^h)_\Omega + (x^h, x^h)_\Omega + (y^h, y^h)_\Gamma \\ &\geqslant 0,\end{aligned} \tag{4.20}$$

where $x^h = \sum_{i=1}^m x_i\varphi_i$ and $y^h = \sum_{i=1}^m y_i\varphi_i$. This shows that B is positive semidefinite. Now, if $X, Y \in \mathbb{R}^M$ satisfy

$$B\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} (A+I)X + BY = 0 \\ -X + M_\Gamma Y = 0, \end{cases}$$

then by multiplying this equality on the left by (X^t, Y^t), we find that $(\nabla x^h, \nabla x^h)_\Omega + (x^h, x^h)_\Omega + (y^h, y^h)_\Gamma = 0$. This implies $X = Y = 0$ which yields that B is invertible, as claimed. Thus, problem (4.14) has a unique maximal solution $(u^h, w^h) \in C^1([0, T^+); V^h \times V^h)$ such that $u^h(0) = u_0^h$. Choosing $\varphi = w^h$ and $\chi = u_t^h$ in (4.14), we find

$$\frac{d}{dt}\mathcal{E}(u^h(t)) + |w^h|_1^2 + |w^h|_0^2 + |u_t^h|_{0,\Gamma}^2 = 0. \tag{4.21}$$

Integrating with respect to t, we deduce (4.18). Using (4.7), equation (4.18) leads to the a priori bound $\|u^h\|_1 \leqslant C(R)$ provided that $u_0^h \in V^h$ with $\|u_0^h\|_1 \leqslant R$. Since $u^h \in L^\infty(0, T^+; H^1(\Omega))$, i.e. the bound is independent of time, we find that the solution is global, i.e. $T^+ = +\infty$, and the proof is complete. □

Exclusively for the next Theorem 4.3.2, we assume that f has a subcritical growth. In other words, we assume that there exists a positive constant c_3 such that

$$|f(s)| \leqslant c_3(1 + |s|^{p-1}),\ \forall s \in \mathbb{R}, \tag{4.22}$$

with $p \in [2, 6]$ when $d = 3$ and $p \geqslant 2$ arbitrary when $d = 2$. When $d = 3$, we also assume that there exists a positive constant c_4 such that

$$|g(s)| \leqslant c_4(1 + |s|^{q-1}),\ \forall s \in \mathbb{R}, \tag{4.23}$$

where $q \geqslant 2$ is arbitrary. The typical choices (4.6) satisfy these assumptions with $p = 4$ and $q = 2$. We have the following theorem:

Theorem 4.3.2. *Assume that $f, g \in C^1(\mathbb{R})$ satisfy (4.5), (4.22) and (4.23). Let $u_0 \in V$ and let $u_0^h \in V^h$ be such that $u_0^h \to u_0$ in V as $h \to 0$. Then, for all $T > 0$, we have*

$$u^h \to u \text{ weak * in } L^\infty(0, T; H_p^1(\Omega)) \text{ and strongly in } C^0([0, T]; L^2(\Omega)),$$

$$(u^h)_{|\Gamma} \to u_{|\Gamma} \text{ weak * in } L^\infty(0, T; H_{per}^1(\Gamma)) \text{ and strongly in } C^0([0, T]; L^2(\Gamma)),$$

$$w^h \to w \text{ weakly in } L^2(0, T; H_p^1(\Omega)),$$

where (u, w) is the unique solution of (4.8) such that $u(0) = u_0$ and

$$u \in L^\infty(0, T; V),\ u_{|\Gamma} \in W^{1,2}(0, T; L^2(\Gamma)) \text{ and } w \in L^2(0, T; H_p^1(\Omega)). \tag{4.24}$$

Proof:
By (4.22) and (4.23), we have that

$$|F(\sigma)| \leqslant c_5|\sigma|^p + c_6 \text{ and } |G(\sigma)| \leqslant c_7|\sigma|^q + c_8, \forall \sigma \in \mathbb{R}, \tag{4.25}$$

where c_5, c_6, c_7 and c_8 are positive constants. Since $u_0^h \to u_0$ in V, using (4.25) and the Sobolev embeddings $H_p^1(\Omega) \subset L^p(\Omega)$ and $H_{per}^1(\Gamma) \subset L^q(\Gamma)$ (with $q = +\infty$ when $d = 2$), we know that $\mathcal{E}(u_0^h)$ is bounded by a constant independent of h. The discrete energy estimate (4.18) implies that $(u^h)_h$ is bounded in $L^\infty(0,T;V)$, $(\nabla w^h)_h$, $(w^h)_h$ are bounded in $L^2(0,T;L^2(\Omega))$ and that $((u^h_{|\Gamma})_t)_h$ is bounded in $L^2(0,T;L^2(\Gamma))$. Thus, we obtain that, up to a subsequence, $u^h \to u$ weak * in $L^\infty(0,T;V)$ and $w^h \to w$ weakly in $L^2(0,T;H^1(\Omega))$.
If $\varphi \in H_p^1(\Omega)$ and $\chi \in V$, then choosing sequences $\varphi^h \in V^h$ and $\chi^h \in V^h$ such that $\varphi^h \to \varphi$ strongly in $H^1(\Omega)$ and $\chi^h \to \chi$ strongly in V. Using standard compactness results, we can pass to the limit in (4.14) and we obtain that (u, w) satisfies (4.1)-(4.4) and (4.24). By uniqueness, the whole sequence (u^h, w^h) converges to (u, w).
For the strong convergence of $(u^h_{|\Gamma})$ to $u_{|\Gamma}$, we use the fact that the space

$$\left\{v \in L^\infty(0,T;H_{per}^1(\Gamma)),\ v_t \in L^2(0,T;L^2(\Gamma))\right\}$$

is compactly embedded into $C^0([0,T];L^2(\Gamma))$. Finally, for the strong convergence of $(u^h)_h$, we use the fact that for all $0 \leqslant s \leqslant t \leqslant T$, we have

$$\begin{aligned}
&|u^h(t) - u^h(s)|_0^2 \\
=&2\int_s^t \left(u_t^h(\sigma), u^h(\sigma) - u^h(s)\right)_\Omega \mathrm{d}\sigma \\
\leqslant& -2\int_s^t \left(\nabla w^h(\sigma), \nabla(u^h(\sigma) - u^h(s))\right)_\Omega \mathrm{d}\sigma - 2\int_s^t \left(w^h(\sigma), u^h(\sigma) - u^h(s)\right)_\Omega \mathrm{d}\sigma \\
\leqslant& 4c\|u^h\|_{L^\infty(0,T;H_p^1(\Omega))} \left(\|\nabla w^h\|_{L^2(0,T;L^2(\Omega))} + \|w^h\|_{L^2(0,T;L^2(\Omega))}\right)|t-s|^{1/2}.
\end{aligned} \tag{4.26}$$

Thus, the sequence $(u^h)_h$ is uniformly equicontinuous in $C^0([0,T];L^2(\Omega))$. Since (u^h) is bounded in $C^0([0,T];H_p^1(\Omega))$ with $H_p^1(\Omega)$ compactly embedded into $L^2(\Omega)$, the Ascoli theorem implies that $u^h \to u$ strongly in $C^0([0,T];L^2(\Omega))$. □

4.4 Error estimates for the space semi-discrete scheme

In order to estimate the errors $u^h - u$ and $w^h - w$ in appropriate norms, we follow a standard approach (see [7], [12] and [55]) and we write

$$\begin{aligned}
&u^h(t) - u(t) = \theta^u(t) + \rho^u(t), \text{ with } \theta^u = u^h - \tilde{u}^h,\ \rho^u = \tilde{u}^h - u, \\
&w^h(t) - w(t) = \theta^w(t) + \rho^w(t), \text{ with } \theta^w = w^h - \tilde{w}^h,\ \rho^w = \tilde{w}^h - w,
\end{aligned}$$

for all $t \in [0,T]$, where $\tilde{u}^h = \tilde{u}^h(t)$ and $\tilde{w}^h = \tilde{w}^h(t)$ are the elliptic projections of $u = u(t)$ and $w = w(t)$, defined by

$$(\nabla\tilde{w}^h, \nabla\chi)_\Omega + (\tilde{w}^h, \chi)_\Omega = (\nabla w, \nabla\chi)_\Omega + (w, \chi)_\Omega,\ \forall \chi \in V^h, \tag{4.27}$$

$$(\nabla\tilde{u}^h,\nabla\chi)_\Omega+(\nabla_\Gamma\tilde{u}^h,\nabla_\Gamma\chi)_\Gamma+\lambda(\tilde{u}^h,\chi)_\Gamma=(\nabla u,\nabla\chi)_\Omega+(\nabla_\Gamma u,\nabla_\Gamma\chi)_\Gamma \\ +\lambda(u,\chi)_\Gamma,\ \forall\chi\in V^h. \tag{4.28}$$

For a given $w\in H^1(\Omega)$, equation (4.27) defines a unique $\tilde{w}^h\in V^h$. Indeed, the bilinear form defined by

$$\tilde{a}(\varphi,\chi)=(\nabla\varphi,\nabla\chi)_\Omega+(\varphi,\chi)_\Omega \tag{4.29}$$

is the scalar product on $H^1(\Omega)$. Thus, applying the Lax-Milgram theorem, we obtain the result. Similarly, for a given $u\in V$, equation (4.28) defines a unique $\tilde{u}^h\in V^h$. Indeed, the norm $v\mapsto|\nabla v|_0^2+|v|_{0,\Gamma}^2$ is equivalent to the H^1-norm, so that the bilinear form defined by

$$a(\varphi,\chi)=(\nabla\varphi,\nabla\chi)_\Omega+(\nabla_\Gamma\varphi,\nabla_\Gamma\chi)_\Gamma+\lambda(\varphi,\chi)_\Gamma \tag{4.30}$$

is coercive on V, i.e. there exists $c_0>0$ such that

$$a(\varphi,\varphi)=(\nabla\varphi,\nabla\varphi)_\Omega+(\nabla_\Gamma\varphi,\nabla_\Gamma\varphi)_\Gamma+\lambda(\varphi,\varphi)_\Gamma\geqslant c_0\|\varphi\|_V^2,\ \forall\varphi\in V.$$

The bilinear continuous form $a(\cdot,\cdot)$ is a fortiori coercive on $V^h\subset V$ and the Lax-Milgram theorem applies.

Lemma 4.4.1. *For all $w\in H^2(\Omega)$, the function $\tilde{w}^h\in V^h$ defined by*

$$(\nabla\tilde{w}^h,\nabla\chi)_\Omega+(\tilde{w}^h,\chi)_\Omega=(\nabla w,\nabla\chi)_\Omega+(w,\chi)_\Omega,\ \forall\chi\in V^h,$$

satisfies

$$|\tilde{w}^h-w|_0+h|\tilde{w}^h-w|_1\leqslant Ch^2|w|_2, \tag{4.31}$$

where C is a positive constant, independent of h.

Proof:
By definition, we have

$$\tilde{a}(\tilde{w}^h,\chi)=\tilde{a}(w,\chi),\quad\forall\chi\in V^h, \tag{4.32}$$

where $\tilde{a}(\cdot,\cdot)$ is defined by (4.29). Since $\tilde{w}^h-I^hw\in V^h$, we have that $\tilde{a}(\tilde{w}^h-w,\tilde{w}^h-I^hw)=0$, which yields

$$\begin{aligned}\tilde{a}(\tilde{w}^h-w,\tilde{w}^h-w)&=\tilde{a}(\tilde{w}^h-w,\tilde{w}^h-I^hw)+\tilde{a}(\tilde{w}^h-w,I^hw-w),\\&=\tilde{a}(\tilde{w}^h-w,I^hw-w),\end{aligned}$$

implying

$$\|\tilde{w}^h-w\|_1^2=\tilde{a}(\tilde{w}^h-w,\tilde{w}^h-w)\leqslant\|\tilde{w}^h-w\|_1\|I^hw-w\|_1.$$

By (4.11), we have

$$\|\tilde{w}^h-w\|_1\leqslant\|I^hw-w\|_1\leqslant Ch|w|_2, \tag{4.33}$$

which gives the H^1-estimate. In order to have the L^2-estimate, we set $\varphi \in H^1(\Omega)$ to be the unique solution of

$$\tilde{a}(\varphi, \chi) = (z, \chi)_\Omega, \quad \forall \chi \in V, \tag{4.34}$$

for a given function $z \in L^2(\Omega)$. Then, $\varphi \in H_p^2(\Omega)$ and thanks to the elliptic regularity, we have

$$|\varphi|_2 \leqslant C|z|_0, \tag{4.35}$$

for some constant $C > 0$ independent of z.
Choosing $\chi = \tilde{w}^h - w$ in (4.34) and using the fact that $\tilde{a}(\tilde{w}^h - w, I^h\varphi) = 0$, we find

$$\begin{aligned}(z, \tilde{w}^h - w)_\Omega =& \tilde{a}(\varphi, \tilde{w}^h - w)\\ =& \tilde{a}(\varphi - I^h\varphi, \tilde{w}^h - w)\\ \leqslant& \|\varphi - I^h\varphi\|_1 \|\tilde{w}^h - w\|_1.\end{aligned} \tag{4.36}$$

Choosing $z = \tilde{w}^h - w$, using (4.11), (4.33) and (4.35), we obtain

$$\begin{aligned}|\tilde{w}^h - w|_0^2 \leqslant& \|\varphi - I^h\varphi\|_1 \|\tilde{w}^h - w\|_1\\ \leqslant& Ch|\varphi|_2 Ch|w|_2\\ \leqslant& Ch^2|\tilde{w}^h - w|_0|w|_2,\end{aligned} \tag{4.37}$$

which gives

$$|\tilde{w}^h - w|_0 \leqslant Ch^2|w|_2. \tag{4.38}$$

By (4.33) and (4.38), we can conclude and the proof is complete. □

Lemma 4.4.2. *For all $u \in H_p^2(\Omega)$ with $u_{|\Gamma} \in H_{per}^2(\Gamma)$ the function $\tilde{u}^h \in V^h$ defined by (4.28) satisfies*

$$|\tilde{u}^h - u|_0 + |\tilde{u}^h - u|_{0,\Gamma} + h|\tilde{u}^h - u|_1 + h|\tilde{u}^h - u|_{1,\Gamma} \leqslant Ch^2(|u|_2 + |u|_{2,\Gamma}), \tag{4.39}$$

where the positive constant C is independent of h.

Proof:
Arguing as above, we have

$$a(\tilde{u}^h, \chi) = a(u, \chi), \quad \forall \chi \in V^h, \tag{4.40}$$

where $a(\cdot, \cdot)$ is defined by (4.30). Since $\tilde{u}^h - I^h u \in V^h$, we have $a(\tilde{u}^h - u, \tilde{u}^h - I^h u) = 0$, which yields

$$\begin{aligned}a(\tilde{u}^h - u, \tilde{u}^h - u) &= a(\tilde{u}^h - u, \tilde{u}^h - I^h u) + a(\tilde{u}^h - u, I^h u - u),\\ &= a(\tilde{u}^h - u, I^h u - u).\end{aligned} \tag{4.41}$$

From (4.41) and using the coercivity of a, we obtain

$$c_0\|\tilde{u}^h - u\|_V^2 \leqslant a(\tilde{u}^h - u, \tilde{u}^h - u) \leqslant c\|\tilde{u}^h - u\|_V \|I^h u - u\|_V.$$

By (4.12), we have

$$\|\tilde{u}^h - u\|_V \leqslant c\|I^h u - u\|_V \leqslant ch(|u|_2 + |u|_{2,\Gamma}), \tag{4.42}$$

which gives the H^1-estimates. In order to have the L^2-estimates, we set $\varphi \in V$ to be the unique solution of

$$a(\varphi, \chi) = (z, \chi)_\Omega + (\psi, \chi)_\Gamma, \quad \forall \chi \in V, \tag{4.43}$$

for some given functions $(z, \psi) \in L^2(\Omega) \times L^2(\Gamma)$. Then, thanks to an elliptic regularity result (see [42]), we have $\varphi \in H^2_p(\Omega),\ \varphi_{|\Gamma} \in H^2_{per}(\Gamma)$ and

$$|\varphi|_2 + |\varphi|_{2,\Gamma} \leqslant C(|z|_0 + |\psi|_{0,\Gamma}), \tag{4.44}$$

for some constant $C > 0$ independent of z and ψ.
Choosing $\chi = \tilde{u}^h - u$ in (4.43) and using the fact that $a(\tilde{u}^h - u, I^h\varphi) = 0$, we find

$$\begin{aligned}(z, \tilde{u}^h - u)_\Omega + (\psi, \tilde{u}^h - u)_\Gamma =& a(\varphi, \tilde{u}^h - u)\\ =& a(\varphi - I^h\varphi, \tilde{u}^h - u)\\ \leqslant& c\|\varphi - I^h\varphi\|_V \|\tilde{u}^h - u\|_V.\end{aligned} \tag{4.45}$$

Choosing $z = \tilde{u}^h - u$ and $\psi = (\tilde{u}^h - u)_\Gamma$, using (4.12), (4.42) and (4.44), we obtain

$$\begin{aligned}|\tilde{u}^h - u|_0^2 + |\tilde{u}^h - u|_{0,\Gamma}^2 \leqslant& c\|\varphi - I^h\varphi\|_1 \|\tilde{u}^h - u\|_1\\ \leqslant& Ch(|\varphi|_2 + |\varphi|_{2,\Gamma}) Ch(|u|_2 + |u|_{2,\Gamma})\\ \leqslant& Ch^2(|\tilde{u}^h - u|_0^2 + |\tilde{u}^h - u|_{0,\Gamma}^2)^{1/2}(|u|_2 + |u|_{2,\Gamma}).\end{aligned} \tag{4.46}$$

Estimate (4.46) leads to

$$(|\tilde{u}^h - u|_0^2 + |\tilde{u}^h - u|_{0,\Gamma}^2)^{1/2} \leqslant Ch^2(|u|_2 + |u|_{2,\Gamma}). \tag{4.47}$$

By (4.42) and (4.47), we can conclude and the proof is complete. □

Lemma 4.4.3. *Let (u, w) the a solution of (4.8) with the initial condition $u(0) = u_0 \in V$ and (u^h, w^h) be the solution of (4.14) with the initial condition $u^h(0) = u_0^h \in V^h$. Assume that*

$$\sup_{t\in[0,T]} \|u(t)\|_{C^0(\overline{\Omega})} < R, \quad \sup_{t\in[0,T]} \|u_t(t)\|_{C^0(\overline{\Omega})} < R, \quad \|u^h(0)\|_{C^0(\overline{\Omega})} < R,$$

for some constant $R < +\infty$, and let $T^h \in (0, T]$ be the maximal time such that $\|u^h(t)\|_{L^\infty(\Omega)} \leqslant R$ for all $t \in [0, T^h]$. Then, the following estimate holds:

$$\begin{aligned}&\mathcal{N}(t) + \int_0^t (|\theta_t^u|_1^2 + \|\theta^w\|_1^2 + \|\theta_t^u\|_{1,\Gamma}^2)\mathrm{d}s\\ \leqslant& C\mathcal{N}(0) + C' \int_0^t (|\rho_t^u|_0^2 + |\rho^u|_0^2 + |\rho^u|_{0,\Gamma}^2 + |\rho_t^u|_{0,\Gamma}^2 + |\rho^w|_0^2 + |\rho_{tt}^u|_0^2 + |\rho_{tt}^u|_{0,\Gamma}^2 + |\rho_t^w|_0^2)\mathrm{d}s,\end{aligned} \tag{4.48}$$

for some positive constants C and C' which are independent of u, u^h and h, where

$$\mathcal{N}(t) = |\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2 + |\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2.$$

Proof:
Subtracting the first equation of (4.8) from the first equation of (4.14), we obtain

$$(u_t^h - u_t, \varphi)_\Omega + (\nabla(w^h - w), \nabla\varphi)_\Omega + (w^h - w, \varphi)_\Omega = 0, \ \forall \varphi \in V^h. \tag{4.49}$$

Using the definitions of θ^u and θ^w as well as (4.27), we find

$$(\theta_t^u, \varphi)_\Omega + (\nabla\theta^w, \nabla\varphi)_\Omega + (\theta^w, \varphi)_\Omega = -(\rho_t^u, \varphi)_\Omega, \ \forall \varphi \in V^h. \tag{4.50}$$

Choosing $\varphi = \theta^w$ in (4.50), we obtain

$$(\theta_t^u, \theta^w)_\Omega + |\theta^w|_1^2 + |\theta^w|_0^2 = -(\rho_t^u, \theta^w)_\Omega. \tag{4.51}$$

Now, subtracting the second equation of (4.8) from the second equation of (4.14) and using (4.28), we find

$$\begin{aligned} &-(\theta^w, \chi)_\Omega + (\nabla\theta^u, \nabla\chi)_\Omega + (\nabla_\Gamma\theta^u, \nabla_\Gamma\chi)_\Gamma + \lambda(\theta^u, \chi)_\Gamma + (\theta_t^u, \chi)_\Gamma \\ &\quad = (\rho^w, \chi)_\Omega - (f(u^h) - f(u), \chi)_\Omega - (g(u^h) - g(u), \chi)_\Gamma - (\rho_t^u, \chi)_\Gamma, \end{aligned} \tag{4.52}$$

for all $\chi \in V^h$.
Choosing $\chi = \theta_t^u$, estimate (4.52) gives

$$\begin{aligned} &-(\theta^w, \theta_t^u)_\Omega + |\theta_t^u|_{0,\Gamma}^2 + \frac{1}{2}\frac{d}{dt}(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2) \\ &\quad = (\rho^w, \theta_t^u)_\Omega - (f(u^h) - f(u), \theta_t^u)_\Omega - (g(u^h) - g(u), \theta_t^u)_\Gamma - (\rho_t^u, \theta_t^u)_\Gamma. \end{aligned} \tag{4.53}$$

Summing (4.51) and (4.53), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2) + |\theta^w|_1^2 + |\theta^w|_0^2 + |\theta_t^u|_{0,\Gamma}^2 \\ &\quad = -(\rho_t^u, \theta^w)_\Omega + (\rho^w, \theta_t^u)_\Omega - (f(u^h) - f(u), \theta_t^u)_\Omega - (g(u^h) - g(u), \theta_t^u)_\Gamma - (\rho_t^u, \theta_t^u)_\Gamma. \end{aligned} \tag{4.54}$$

We have:

$$\begin{aligned} |f(u^h) - f(u)|_0 &\leqslant L_f|u^h - u|_0, \\ |g(u^h) - g(u)|_{0,\Gamma} &\leqslant L_g|u^h - u|_{0,\Gamma}, \end{aligned} \tag{4.55}$$

on $[0, T^h]$, where L_f and L_g are respectively the Lipschitz constants of f and g on $[-R, R]$. Thus, using (4.55) and the Hölder inequality, estimate (4.54) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2) + |\theta^w|_1^2 + |\theta^w|_0^2 + |\theta_t^u|_{0,\Gamma}^2 \\ \leqslant &|\rho_t^u|_0|\theta^w|_0 + |\rho^w|_0|\theta_t^u|_0 + L_f(|\theta^u|_0 + |\rho^u|_0)|\theta_t^u|_0 \\ &+ L_g(|\theta^u|_{0,\Gamma} + |\rho^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma} + |\rho_t^u|_{0,\Gamma}|\theta_t^u|_{0,\Gamma}. \end{aligned} \tag{4.56}$$

Using the inequality

$$ab \leqslant \epsilon a^2 + 1/(4\epsilon)b^2, \quad \forall a, b \geqslant 0, \quad \forall \epsilon > 0,$$

with ε conveniently chosen, estimate (4.56) gives

$$\begin{aligned}&\frac{d}{dt}(|\theta^u|_1^2+\lambda|\theta^u|_{0,\Gamma}^2+|\nabla_\Gamma\theta^u|_{0,\Gamma}^2)+|\theta^w|_1^2+|\theta^w|_0^2+|\theta_t^u|_{0,\Gamma}^2\\&\leqslant C_1(|\rho_t^u|_0^2+|\rho^u|_0^2+|\rho^u|_{0,\Gamma}^2+|\rho_t^u|_{0,\Gamma}^2+|\rho^w|_0^2)+C_2(|\theta^u|_{0,\Gamma}^2+|\theta^u|_0^2+|\theta_t^u|_0^2),\end{aligned}\tag{4.57}$$

for some positive constants C_1 and C_2 which depend on $|\Omega|, |\Gamma|, L_f$ and L_g.
To estimate θ_t^u, we differentiate (4.50) and (4.52) with respect to t. We obtain

$$(\theta_{tt}^u,\varphi)_\Omega+(\nabla\theta_t^w,\nabla\varphi)_\Omega+(\theta_t^w,\varphi)_\Omega=-(\rho_{tt}^u,\varphi)_\Omega,\forall\varphi\in V^h\tag{4.58}$$

and

$$\begin{aligned}&-(\theta_t^w,\chi)_\Omega+(\nabla\theta_t^u,\nabla\chi)_\Omega+(\nabla_\Gamma\theta_t^u,\nabla_\Gamma\chi)_\Gamma+\lambda(\theta_t^u,\chi)_\Gamma+(\theta_{tt}^u,\chi)_\Gamma\\&\quad=(\rho_t^w,\chi)_\Omega-([f(u^h)-f(u)]_t,\chi)_\Omega-([g(u^h)-g(u)]_t,\chi)_\Gamma-(\rho_{tt}^u,\chi)_\Gamma,\forall\chi\in V^h.\end{aligned}\tag{4.59}$$

Choosing $\varphi=G^h\theta_t^u$ in (4.58) and $\chi=\theta_t^u$ in (4.59), adding the resulting equations and using the fact that

$$(\nabla\theta_t^w,\nabla G^h\theta_t^u)_\Omega+(\theta_t^w,G^h\theta_t^u)_\Omega=(\theta_t^w,\theta_t^u)_\Omega,$$

we find

$$\begin{aligned}&\frac12\frac{d}{dt}(|\theta_t^u|_{-1,h}^2+|\theta_t^u|_{0,\Gamma}^2)+|\theta_t^u|_1^2+|\theta_t^u|_{1,\Gamma}^2+\lambda|\theta_t^u|_{0,\Gamma}^2\\&=-(\rho_{tt}^u,G^h\theta_t^u)_\Omega+(\rho_t^w,\theta_t^u)_\Omega-([f(u^h)-f(u)]_t,\theta_t^u)_\Omega-([g(u^h)-g(u)]_t,\theta_t^u)_\Gamma\\&\quad-(\rho_{tt}^u,\theta_t^u)_\Gamma.\end{aligned}\tag{4.60}$$

Employing

$$[f(u^h)-f(u)]_t=f'(u^h)[u_t^h-u_t]+[f'(u^h)-f'(u)]u_t$$

and

$$[g(u^h)-g(u)]_t=g'(u^h)[u_t^h-u_t]+[g'(u^h)-g'(u)]u_t,$$

we find

$$\begin{aligned}([f(u^h)-f(u)]_t,\theta_t^u)_\Omega=&(f'(u^h)[u_t^h-u_t],\theta_t^u)_\Omega+([f'(u^h)-f'(u)]u_t,\theta_t^u)_\Omega\\\leqslant&\sup_{[-R,R]}|f'|(|\theta_t^u|_0+|\rho_t^u|_0)|\theta_t^u|_0+RL_{f'}(|\theta^u|_0+|\rho^u|_0)|\theta_t^u|_0\end{aligned}$$

and

$$\begin{aligned}([g(u^h)-g(u)]_t,\theta_t^u)_\Gamma=&(g'(u^h)[u_t^h-u_t],\theta_t^u)_\Gamma+([g'(u^h)-g'(u)]u_t,\theta_t^u)_\Gamma\\\leqslant&\sup_{[-R,R]}|g'|(|\theta_t^u|_{0,\Gamma}+|\rho_t^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma}+RL_{g'}(|\theta^u|_{0,\Gamma}+|\rho^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma},\end{aligned}$$

where $L_{f'}$ and $L_{g'}$ are respectively the Lipschitz constants of f' and g' on $[-R,R]$. Thus, (4.60) implies

$$\begin{aligned}&\frac12\frac{d}{dt}(|\theta_t^u|_{-1,h}^2+|\theta_t^u|_{0,\Gamma}^2)+|\theta_t^u|_1^2+|\theta_t^u|_{1,\Gamma}^2+\lambda|\theta_t^u|_{0,\Gamma}^2\\\leqslant&|\rho_{tt}^u|_{-1,h}|\theta_t^u|_{-1,h}+|\rho_{tt}^u|_{0,\Gamma}|\theta_t^u|_{0,\Gamma}+|\rho_t^w|_0|\theta_t^u|_0\\&+\sup_{[-R,R]}|f'|(|\theta_t^u|_0+|\rho_t^u|_0)|\theta_t^u|_0+RL_{f'}(|\theta^u|_0+|\rho^u|_0)|\theta_t^u|_0\\&+\sup_{[-R,R]}|g'|(|\theta_t^u|_{0,\Gamma}+|\rho_t^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma}+RL_{g'}(|\theta^u|_{0,\Gamma}+|\rho^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma}.\end{aligned}\tag{4.61}$$

Using (4.17), the fact that the norm $v \mapsto |\nabla v|_0^2 + |v|_{0,\Gamma}^2$ is equivalent to the H^1-norm and the interpolation property (4.16) applied to θ_t^u, (4.61) gives

$$\begin{aligned}
&\frac{d}{dt}(|\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2) + |\theta_t^u|_1^2 + |\theta_t^u|_{1,\Gamma}^2 + \lambda|\theta_t^u|_{0,\Gamma}^2 \\
&\quad \leqslant C_3(|\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2 + |\theta^u|_0^2 + |\theta^u|_{0,\Gamma}^2) \\
&\quad + C_4(|\rho_{tt}^u|_0^2 + |\rho_{tt}^u|_{0,\Gamma}^2 + |\rho_t^w|_0^2 + |\rho_t^u|_0^2 + |\rho^u|_0^2 + |\rho_t^u|_{0,\Gamma}^2 + |\rho^u|_{0,\Gamma}^2), \text{ on } (0, T^h),
\end{aligned} \tag{4.62}$$

for some positive constants C_3 and C_4 which depend on $\lambda, R, c_p, L_{f'}, L_{g'}, \sup_{[-R,R]} |f'|$ and $\sup_{[-R,R]} |f'|$. Adding (4.57) and (4.62), we find

$$\begin{aligned}
&\frac{d}{dt}(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2 + |\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2) + |\theta^w|_1^2 + |\theta^w|_0^2 + |\theta_t^u|_{0,\Gamma}^2 + |\theta_t^u|_1^2 + |\theta_t^u|_{1,\Gamma}^2 \\
&\leqslant C_5(|\rho_t^u|_0^2 + |\rho^u|_0^2 + |\rho^u|_{0,\Gamma}^2 + |\rho_t^u|_{0,\Gamma}^2 + |\rho^w|_0^2 + |\rho_{tt}^u|_0^2 + |\rho_{tt}^u|_{0,\Gamma}^2 + |\rho_t^w|_0^2) \\
&+ C_6(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2 + |\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2 + |\theta_t^u|_0^2),
\end{aligned} \tag{4.63}$$

where we use the fact that $|\theta^u|_0^2 \leqslant c(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2)$ and for θ_t^u we use again the interpolation property (4.16). Applying Gronwall's lemma, we find estimate (4.48) with $C = e^{C_6 T}$ and $C' = C_5 e^{C_6 T}$. □

Theorem 4.4.4. *Let (u, w) be the solution of problem (4.1)-(4.4) with the initial condition $u(0) = u_0$ such that*

$$u, u_t, u_{tt}, w, w_t \in L^2(0, T; H_p^2(\Omega)) \tag{4.64}$$

and

$$u_{|\Gamma}, (u_t)_{|\Gamma}, (u_{tt})_{|\Gamma}, w_{|\Gamma}, (w_t)_{|\Gamma} \in L^2(0, T; H_{per}^2(\Gamma)) \tag{4.65}$$

and let (u^h, w^h) be the solution of problem (4.14) with initial condition $u^h(0) = u_0^h$. If

$$\theta^u(0) = 0 \text{ and } \theta^w(0) = 0, \tag{4.66}$$

then, the following estimates hold:

$$\begin{aligned}
\sup_{[0,T]} \left(|u^h - u|_0 + |u^h - u|_{0,\Gamma} + |u_t^h - u_t|_{-1,h} + |u_t^h - u_t|_{0,\Gamma}\right) &\leqslant Ch^2, \\
\int_0^T |w^h - w|_0^2 \mathrm{d}s &\leqslant Ch^4, \\
\sup_{[0,T]} \left(|u^h - u|_1 + |u^h - u|_{1,\Gamma}\right) &\leqslant Ch, \\
\int_0^T \left(|w^h - w|_1^2 + |u_t^h - u_t|_1^2 + |u_t^h - u_t|_{1,\Gamma}^2\right) \mathrm{d}s &\leqslant Ch^2.
\end{aligned}$$

Proof:
If we differentiate equations (4.27) and (4.28) with respect to t, we obtain that the elliptic projections of u_t and w_t are respectively $(\tilde{u})_t$ and $(\tilde{w})_t$. A similar statement

holds for u_{tt}. Therefore, Lemma 4.4.1 applies with w replaced by w_t and Lemma 4.4.2 applies with u replaced by u_t, u_{tt}, i.e.

$$\begin{aligned} |\rho_t^u|_0 + |\rho_t^u|_{0,\Gamma} + h|\rho_t^u|_1 + h|\rho_t^u|_{1,\Gamma} &\leqslant Ch^2(|u_t|_2 + |u_t|_{2,\Gamma}), \\ |\rho_{tt}^u|_0 + |\rho_{tt}^u|_{0,\Gamma} + h|\rho_{tt}^u|_1 + h|\rho_{tt}^u|_{1,\Gamma} &\leqslant Ch^2(|u_{tt}|_2 + |u_{tt}|_{2,\Gamma}), \\ |\rho_t^w|_0 + h|\rho_t^w|_1 &\leqslant Ch^2|w_t|_2. \end{aligned} \tag{4.67}$$

The regularity required on u, u_t implies that $u \in C^1([0,T]; H_p^2(\Omega))$ and by the Sobolev continuous injection $H_p^2(\Omega) \subset C^0(\overline{\Omega})$, we see that u and u_t belong to $C^0([0,T]; C^0(\overline{\Omega}))$. Thus,

$$\sup_{t\in[0,T]} \|u(t)\|_{C^0(\overline{\Omega})} < R \text{ and } \sup_{t\in[0,T]} \|u_t(t)\|_{C^0(\overline{\Omega})} < R,$$

for some $R > 0$. We also have

$$\begin{aligned} \|u_0^h - u_0\|_{C^0(\overline{\Omega})} &\leqslant \|u_0^h - I^h u_0\|_{C^0(\overline{\Omega})} + \|I^h u_0 - u_0\|_{C^0(\overline{\Omega})} \\ &\leqslant Ch^{-d/2}|u_0^h - I^h u_0|_0 + \|I^h u_0 - u_0\|_{C^0(\overline{\Omega})} \text{ (using (4.13))} \\ &\leqslant Ch^{-d/2}(|u_0^h - u_0|_0 + |u_0 - I^h u_0|_0) + \|I^h u_0 - u_0\|_{C^0(\overline{\Omega})}. \end{aligned} \tag{4.68}$$

Using the embedding that $H_p^2(\Omega) \subset C^{0,\gamma}(\Omega)$, where $\gamma \in (0,1)$, we find

$$\|I^h u_0 - u_0\|_{C^0(\overline{\Omega})} \leqslant C' h^\gamma |u_0|_2. \tag{4.69}$$

Due to estimates (4.11), (4.39) and assumptions (4.66), we have

$$|u_0^h - u_0|_0 + |u_0 - I^h u_0|_0 \leqslant Ch^2(|u_0|_2 + |u_0|_{2,\Gamma}). \tag{4.70}$$

Thus by (4.69) and (4.70), we deduce that

$$\|u_0^h - u_0\|_{C^0(\overline{\Omega})} \leqslant (Ch^{2-d/2} + C'h^\gamma)|u_0|_2 + Ch^{2-d/2}|u_0|_{2,\Gamma}. \tag{4.71}$$

For h small enough, we obtain

$$\|u_0^h\|_{C^0(\overline{\Omega})} < R, \tag{4.72}$$

and we may apply Lemma (4.4.3).
We claim that $\mathcal{N}(0) \leqslant Ch^4$, with $\mathcal{N}(t) = |\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\nabla_\Gamma \theta^u|_{0,\Gamma}^2 + |\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2$. Indeed, by assumptions (4.66), we have

$$\mathcal{N}(0) = |\theta_t^u(0)|_{-1,h}^2 + |\theta_t^u(0)|_{0,\Gamma}^2. \tag{4.73}$$

Using the fact that $\theta^w(0) = 0$ and estimate (4.50) at $t = 0$, we see that $u^h(0)$ satisfies

$$(\theta_t^u(0), \varphi)_\Omega = -(\rho_t^u(0), \varphi)_\Omega, \ \forall \varphi \in V^h. \tag{4.74}$$

Choosing $\varphi = G^h \theta_t^u(0)$ in (4.74) and using (4.17) and (4.67), we obtain

$$\begin{aligned} |\theta_t^u(0)|_{-1,h}^2 &= -(\rho_t^u(0), G^h \theta_t^u(0))_\Omega \\ &\leqslant |\rho_t^u(0)|_{-1,h} |\theta_t^u(0)|_{-1,h} \\ &\leqslant c|\rho_t^u(0)|_0 |\theta_t^u(0)|_{-1,h} \\ &\leqslant Ch^2(|u_t(0)|_2 + |u_t(0)|_{2,\Gamma})|\theta_t^u(0)|_{-1,h}. \end{aligned} \tag{4.75}$$

For $\varphi = \theta_t^u(0)$ in (4.74), we find

$$|\theta_t^u(0)|_0 \leqslant |\rho_t^u(0)|_0 \leqslant Ch^2(|u_t(0)|_2 + |u_t(0)|_{2,\Gamma}). \tag{4.76}$$

Using assumptions (4.66) and choosing $\chi = \theta_t^u(0)$ in (4.52) considered at $t = 0$, we obtain

$$\begin{aligned} |\theta_t^u(0)|_{0,\Gamma}^2 =&(\rho^w(0), \theta_t^u(0))_\Omega - (f(u_0^h) - f(u_0), \theta_t^u(0))_\Omega \\ &- (g(u_0^h) - g(u_0), \theta_t^u(0))_\Gamma - (\rho_t^u(0), \theta_t^u(0))_\Gamma. \end{aligned} \tag{4.77}$$

Using the fact that $\|u_0\|_{C^0(\overline{\Omega})} < R$ and the fact that $u_0^h - u_0 = \rho^u(0)$, (4.77) gives

$$\begin{aligned} |\theta_t^u(0)|_{0,\Gamma}^2 \leqslant&(|\rho^w(0)|_0 + L_f|\rho^u(0)|_0)|\theta_t^u(0)|_0 \\ &+ (L_g|\rho^u(0)|_{0,\Gamma} + |\rho_t^u(0)|_{0,\Gamma})|\theta_t^u(0)|_{0,\Gamma}, \end{aligned} \tag{4.78}$$

where L_f and L_g are respectively the Lipschitz constants of f and g on $[-R, R]$. By (4.31), (4.39), (4.67) and (4.76), estimate (4.78) yields

$$\begin{aligned} |\theta_t^u(0)|_{0,\Gamma}^2 \leqslant&Ch^2(|w(0)|_2 + |u_0|_2 + |u_0|_{2,\Gamma})Ch^2(|u_t(0)|_2 + |u_t(0)|_{2,\Gamma}) \\ &+ (L_gCh^2(|u_0|_2 + |u_0|_{2,\Gamma}) + Ch^2(|u_t(0)|_2 + |u_t(0)|_{2,\Gamma}))|\theta_t^u(0)|_{0,\Gamma}, \end{aligned} \tag{4.79}$$

and

$$|\theta_t^u(0)|_{0,\Gamma}^2 \leqslant Ch^4 + Ch^2|\theta_t^u(0)|_{0,\Gamma}. \tag{4.80}$$

In particular, (4.80) implies $|\theta_t^u(0)|_{0,\Gamma} \leqslant Ch^2$. Thus,

$$\mathcal{N}(0) \leqslant Ch^4. \tag{4.81}$$

The regularity assumptions on u and w and estimates (4.31), (4.39) and (4.67), imply that

$$\int_0^t |\rho_t^u|_0^2 + |\rho^w|_0^2 + |\rho^u|_0^2 + |\rho_{tt}^u|_0^2 \mathrm{d}s \leqslant Ch^4. \tag{4.82}$$

Using estimates (4.81) and (4.82) we deduce from (4.48) that

$$\mathcal{N}(t) \leqslant Ch^4, \quad \forall t \in [0, T^h]. \tag{4.83}$$

Estimate (4.83) implies in particular that

$$|\theta^u(t)|_0^2 \leqslant Ch^2, \quad \forall t \in [0, T^h]. \tag{4.84}$$

Arguing as in (4.68), we deduce that

$$\sup_{t\in[0,T^h]} \|u^h(t) - u(t)\|_{C^0(\overline{\Omega})} \to 0 \text{ as } h \to 0. \tag{4.85}$$

We conclude by noticing that for h small enough, $T^h = T$. □

Remark 4.4.5. We remark here that the regularity required in (4.64) and (4.65) is a strong one, this is due to the fact that we need strong regularity results in order to estimate the term θ_t^u.

4.5 Stability of the backward Euler scheme

In what follows, we denote by $\delta t = T/N$ the time step with $N \in \mathbb{N}^*$. We study the following backward in time Euler scheme:
Let $u_h^0 \in V^h$ and for $n = 1, 2, ...$, find $(u_h^n, w_h^n) \in V^h \times V^h$ such that

$$\begin{cases} (\overline{\partial} u_h^n, \varphi)_\Omega = -(\nabla w_h^n, \nabla \varphi)_\Omega - (w_h^n, \varphi)_\Omega, \\ (w_h^n, \chi)_\Omega = (\nabla u_h^n, \nabla \chi)_\Omega + (f(u_h^n), \chi)_\Omega + (\nabla_\Gamma u_h^n, \nabla_\Gamma \chi)_\Gamma + (\tilde{g}(u_h^n), \chi)_\Gamma + (\overline{\partial} u_h^n, \chi)_\Gamma, \end{cases} \tag{4.86}$$

for all $\varphi, \chi \in V^h$, where we denote by $\overline{\partial}$ the operator which to a sequence $(v^n)_{n \geqslant 0}$ associates the sequence defined by

$$\overline{\partial} v^n = \frac{v^n - v^{n-1}}{\delta t}, \ n \geqslant 1, \tag{4.87}$$

and the function $\tilde{g}$ is given by

$$\tilde{g}(\sigma) = \lambda \sigma + g(\sigma), \quad \forall \sigma \in \mathbb{R}.$$

Note that the dissipativity assumptions (4.5) imply that

$$f'(v) \geqslant -C_f \text{ and } \tilde{g}'(v) \geqslant -C_g, \quad \forall v \in \mathbb{R}, \tag{4.88}$$

where C_f and C_g are positive constants.

Theorem 4.5.1. *For every $u_h^0 \in V^h$, there exists a sequence $(u_h^n, w_h^n)_{n \geqslant 1}$ generated by* (4.86) *and which satisfies*

$$\mathcal{E}(u_h^n) + \frac{1}{2\delta t}|u_h^n - u_h^{n-1}|_{-1,h}^2 + \frac{1}{2\delta t}|u_h^n - u_h^{n-1}|_{0,\Gamma}^2 \leqslant \mathcal{E}(u_h^{n-1}), \ \forall n \geqslant 1. \tag{4.89}$$

Furthermore, if $\delta t < \delta t^*$, where $\delta t^* = \min\left\{ \dfrac{4}{C_f^2 + C_f^2 C}, \dfrac{1}{C_g} \right\}$, then this sequence is uniquely defined.

Proof:
Consider the variational problem:

$$J^h(u) = \inf_{v \in V^h} J^h(v), \tag{4.90}$$

where

$$J^h(v) = \mathcal{E}(v) + \frac{1}{2\delta t}|v - u_h^{n-1}|_{-1,h}^2 + \frac{1}{2\delta t}|v - u_h^{n-1}|_{0,\Gamma}^2. \tag{4.91}$$

Using (4.7), we have

$$J^h(v) \geqslant \frac{1}{2}(|v|_1^2 + |v|_{1,\Gamma}^2 + \lambda |v|_{0,\Gamma}^2) - c_2(|\Omega| + |\Gamma|), \quad \forall v \in V^h. \tag{4.92}$$

Since J^h is continuous, there exists a solution to (4.90). Such a solution u satisfies

$$\begin{aligned} 0 =& (\nabla u, \nabla \chi)_\Omega + (f(u), \chi)_\Omega + (\nabla_\Gamma u, \nabla_\Gamma \chi)_\Gamma + (\tilde{g}(u), \chi)_\Gamma \\ &+ \frac{1}{\delta t}(G^h(u - u_h^{n-1}), \chi)_\Omega + \frac{1}{\delta t}(u - u_h^{n-1}, \chi)_\Gamma, \ \forall \chi \in V^h. \end{aligned} \tag{4.93}$$

Setting $u_h^n = u,\ w_h^n = -\frac{1}{\delta t}G^h(u - u_h^{n-1})$, we find that (u_h^n, w_h^n) solves problem (4.86). Thanks to (4.90), $J^h(u_h^n) \leqslant J^h(u_h^{n-1})$, which implies (4.89).
In order to prove uniqueness, set $\xi = (u_h^n)^1 - (u_h^n)^2$ and $\eta = (w_h^n)^1 - (w_h^n)^2$ to be the difference of two possible solutions $((u_h^n)^i, (w_h^n)^i)$ $(i = 1, 2)$ of (4.86) for a given u_h^{n-1}. Then, (ξ, η) satisfies

$$\begin{cases} (\xi, \varphi)_\Omega = -\delta t(\nabla\eta, \nabla\varphi)_\Omega - \delta t(\eta, \varphi)_\Omega, \\ (\eta, \chi)_\Omega = (\nabla\xi, \nabla\chi)_\Omega + (f((u_h^n)^1) - f((u_h^n)^2), \chi)_\Omega + (\nabla_\Gamma\xi, \nabla_\Gamma\chi)_\Gamma \\ \qquad\qquad + \left(\tilde{g}((u_h^n)^1) - \tilde{g}((u_h^n)^2), \chi\right)_\Gamma + (\xi/\delta t, \chi)_\Gamma, \end{cases} \tag{4.94}$$

for all $\varphi, \chi \in V^h$. Taking $\varphi = \eta$ in the first equation of (4.94) and $\chi = \xi$ in the second equation of (4.94) and subtracting the resulting equations, we obtain

$$\delta t|\eta|_1^2 + \delta t|\eta|_0^2 + |\xi|_1^2 + |\xi|_{1,\Gamma}^2 + \frac{1}{\delta t}|\xi|_{0,\Gamma}^2 \leqslant C_f|\xi|_0^2 + C_g|\xi|_{0,\Gamma}^2, \tag{4.95}$$

where we have used the inequalities

$$(f((u_h^n)^1) - f((u_h^n)^2), \xi)_\Omega \geqslant -C_f|\xi|_0^2,$$

$$(\tilde{g}((u_h^n)^1) - \tilde{g}((u_h^n)^2), \xi)_\Gamma \geqslant -C_g|\xi|_{0,\Gamma}^2.$$

By choosing $\chi = \xi$ in the first equation of (4.94), we find

$$\begin{aligned} C_f|\xi|_0^2 =& -C_f\delta t(\nabla\eta, \nabla\xi)_\Omega - C_f\delta t(\eta, \xi)_\Omega \\ \leqslant& \delta t|\eta|_1^2 + \frac{C_f^2\delta t}{4}|\xi|_1^2 + \delta t|\eta|_0^2 + \frac{C_f^2\delta t}{4}|\xi|_0^2, \\ \leqslant& \delta t|\eta|_1^2 + \frac{C_f^2\delta t}{4}|\xi|_1^2 + \delta t|\eta|_0^2 + \frac{C_f^2 C\delta t}{4}|\xi|_1^2, \end{aligned} \tag{4.96}$$

which leads to

$$\left(1 - \delta t(\frac{C_f^2}{4} + \frac{C_f^2 C}{4})\right)|\xi|_1^2 + |\xi|_{1,\Gamma}^2 + \left(\frac{1}{\delta t} - C_g\right)|\xi|_{0,\Gamma}^2 \leqslant 0. \tag{4.97}$$

The smallness assumption on δt implies $\xi = 0$ and by (4.94), we deduce that $\eta = 0$. □

Corollary 4.5.2. *If f and g are analytic, then, for all $u_h^0 \in V^h$, any sequence $(u_h^n, w_h^n)_{n\geqslant 1}$ generated by* (4.86) *and which satisfies the energy estimate* (4.89) *converges to a steady state $(\overline{u}_h, \overline{w}_h)$ as $n \to +\infty$.*

Proof:
Let $u_h^0 \in V^h$. By (4.89), the sequence $(\mathcal{E}(u_h^n))_n$ is non-increasing and since it is bounded from below by 0, we have $\mathcal{E}(u_h^n) \to \mathcal{E}^*$. We assume without loss of generality that $\mathcal{E}^* = 0$. By (4.9), $\mathcal{E}(v) \to +\infty$ as $\|v\|_V \to +\infty$ and $(u_h^n)_n$ is bounded: there exist $u_h^\infty \in V^h$ and a subsequence $(u_h^{n_k})_k$ such that $u_h^{n_k} \to u_h^\infty$ in V^h as $k \to +\infty$.
Using the same matrix notation as introduced in the proof of Proposition 4.3.1, problem (4.86) reads

$$\begin{pmatrix} (A+I) & I \\ -I & M_\Gamma \end{pmatrix} \begin{pmatrix} W^n \\ (U^n - U^{n-1})/\delta t \end{pmatrix} = -\begin{pmatrix} 0 \\ AU^n + F^h(U^n) + A_\Gamma U^n + G_\Gamma^h(U^n) \end{pmatrix},$$

where U^n (resp. W^n) is the vector of the coordinates of u_h^n (resp. w_h^n). The matrix $A+I$ is invertible. Thus, eliminating W^n, we obtain

$$((A+I)^{-1}+M_\Gamma)\frac{U^n-U^{n-1}}{\delta t} = -(AU^n+F^h(U^n)+A_\Gamma U^n+G_\Gamma^h(U^n)) = -\nabla E^h(U^n), \tag{4.98}$$

where

$$E^h(V) = \mathcal{E}(\sum_{i=1}^{M} v_i\varphi_i), \quad \forall V=(v_1,\cdots,v_M)\in \mathbb{R}^M.$$

If we take the Euclidean norm of (4.98), we see that

$$\lambda_1\frac{\|U^n-U^{n-1}\|}{\delta t} \leqslant \|\nabla E^h(U^n)\| \leqslant \lambda_M\frac{\|U^n-U^{n-1}\|}{\delta t}, \tag{4.99}$$

where $0\leqslant\lambda_1<\lambda_M<+\infty$ are respectively the smallest and the largest eigenvalues of $((A+I)^{-1}+M_\Gamma)$ since $((A+I)^{-1}+M_\Gamma)$ is a symmetric positive definite matrix. On the other hand, since f and g are real analytic, the function E^h is real analytic on $\mathbb{R}^M$ and it satisfies the Lojasiewicz inequality; more precisely, there exist $\sigma,\gamma>0$ and $\nu\in(0,1/2]$ such that

$$\forall V\in\mathbb{R}^M, \|V-U^\infty\|<\sigma \Rightarrow |E^h(V)|^{1-\nu}\leqslant\gamma\|\nabla E^h(V)\|, \tag{4.100}$$

where we have used the fact that $E^h(U^\infty)=\mathcal{E}(u_h^\infty)=\mathcal{E}^*=0$ and where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^M$. Now let n be such that $\|U^n-U^\infty\|\leqslant\sigma$. We recall that E^h satisfies the following inequality:

$$E^h(U^n)+\frac{1}{2\delta t}|U^n-U^{n-1}|_{-1,h}^2+\frac{1}{2\delta t}|U^n-U^{n-1}|_{0,\Gamma}^2\leqslant E^h(U^{n-1}),\ \forall n\geqslant 1. \tag{4.101}$$

We consider the following two cases:

Case 1: $E^h(U^n)>E^h(U^{n-1})/2$. In what follows, we will use the fact that all norms are equivalent on V^h. Since $x\mapsto x^{\nu-1}$ is non-increasing, we have

$$2E^h(U^n)>E^h(U^{n-1}) \Rightarrow E^h(U^{n-1})^{\nu-1}>2^{\nu-1}E^h(U^n)^{\nu-1}. \tag{4.102}$$

We also know that

$$|v^h|_{-1,h}^2+|v^h|_{0,\Gamma}^2\geqslant c_h|v^h|_0^2, \quad \forall v^h\in V^h \tag{4.103}$$

for some positive constant $c_h>0$ since all norms are equivalent on V^h. Then

$$\begin{aligned}
E^h(U^{n-1})^\nu-E^h(U^n)^\nu &= \int_{E^h(U^n)}^{E^h(U^{n-1})}\nu x^{\nu-1}\mathrm{d}x\\
&\geqslant \int_{E^h(U^n)}^{E^h(U^{n-1})}\nu(E^h(U^{n-1}))^{\nu-1}\mathrm{d}x \text{ (using (4.102))}\\
&\geqslant 2^{\nu-1}\nu E^h(U^n)^{\nu-1}[E^h(U^{n-1})-E^h(U^n)] \text{ (using (4.101))}\\
&\geqslant 2^{\nu-1}\nu E^h(U^n)^{\nu-1}\frac{1}{2\delta t}(|U^n-U^{n-1}|_{-1,h}^2+|U^n-U^{n-1}|_{0,\Gamma}^2)\\
&= 2^{\nu-2}\nu E^h(U^n)^{\nu-1}\frac{1}{\delta t}(|U^n-U^{n-1}|_{-1,h}^2+|U^n-U^{n-1}|_{0,\Gamma}^2)\\
&\geqslant 2^{\nu-2}\nu c_h\frac{\|U^n-U^{n-1}\|^2}{\delta t E^h(U^n)^{1-\nu}}.
\end{aligned} \tag{4.104}$$

Using (4.100) and (4.99), we obtain

$$\begin{aligned} E^h(U^{n-1})^\nu - E^h(U^n)^\nu \geqslant & \frac{2^{\nu-2}\nu c_h}{\gamma \delta t} \frac{\|U^n - U^{n-1}\|^2}{\|\nabla E^h(U^n)\|} \\ \geqslant & \frac{2^{\nu-2}\nu c_h}{\lambda_M \gamma} \|U^n - U^{n-1}\|. \end{aligned} \tag{4.105}$$

Case 2: $E^h(U^n) \leqslant E^h(U^{n-1})/2$. We have

$$\begin{aligned} & E^h(U^n) \leqslant \frac{E^h(U^{n-1})}{2} \\ \Longleftrightarrow & (E^h(U^n))^{1/2} \leqslant \frac{(E^h(U^{n-1}))^{1/2}}{\sqrt{2}} \\ \Longleftrightarrow & (E^h(U^{n-1}))^{1/2} - \frac{(E^h(U^{n-1}))^{1/2}}{\sqrt{2}} \leqslant (E^h(U^{n-1}))^{1/2} - (E^h(U^n))^{1/2} \\ \Longleftrightarrow & \left(1 - \frac{1}{\sqrt{2}}\right) (E^h(U^{n-1}))^{1/2} \leqslant (E^h(U^{n-1}))^{1/2} - (E^h(U^n))^{1/2} \\ \Longleftrightarrow & (E^h(U^{n-1}))^{1/2} \leqslant \left(1 - \frac{1}{\sqrt{2}}\right)^{-1} \left((E^h(U^{n-1}))^{1/2} - (E^h(U^n))^{1/2}\right). \end{aligned} \tag{4.106}$$

Using (4.101), (4.103) and (4.106), we obtain

$$\begin{aligned} \|U^n - U^{n-1}\| \leqslant & \frac{1}{\sqrt{c_h}} (|U^n - U^{n-1}|_{-1,h} + |U^n - U^{n-1}|_{0,\Gamma}) \\ \leqslant & 2 \left(\frac{\delta t}{c_h}\right)^{1/2} \left(E^h(U^{n-1}) - E^h(U^n)\right)^{1/2} \\ \leqslant & 2 \left(\frac{\delta t}{c_h}\right)^{1/2} E^h(U^{n-1})^{1/2} \\ \leqslant & 2 \left(1 - \frac{1}{\sqrt{2}}\right)^{-1} \left(\frac{\delta t}{c_h}\right)^{1/2} \left(E^h(U^{n-1})^{1/2} - E^h(U^n)^{1/2}\right). \end{aligned} \tag{4.107}$$

Thus, in both cases, we have

$$\begin{aligned} \|U^n - U^{n-1}\| \leqslant & \frac{2^{2-\nu}\lambda_M \gamma}{\nu c_h} \left(E^h(U^{n-1})^\nu - E^h(U^n)^\nu\right) \\ & + 2 \left(1 - \frac{1}{\sqrt{2}}\right)^{-1} \left(\frac{\delta t}{c_h}\right)^{1/2} \left(E^h(U^{n-1})^{1/2} - E^h(U^n)^{1/2}\right) \\ \leqslant & \frac{2^{2-\nu}\lambda_M \gamma}{\nu c_h} \left(E^h(U^{n-1})^\nu - E^h(U^n)^\nu\right) \\ & + \left(\frac{\delta t}{c_h}\right)^{1/2} \left(E^h(U^{n-1})^{1/2} - E^h(U^n)^{1/2}\right). \end{aligned} \tag{4.108}$$

Now, let $\tilde{E} > 0$ be small enough so that

$$\frac{2^{2-\nu}\lambda_M \gamma}{\nu c_h} \tilde{E}^\nu + \left(\frac{\delta t}{c_h}\right)^{1/2} \tilde{E}^{1/2} \leqslant \sigma/3. \tag{4.109}$$

We choose $\overline{n}$ large enough such that
$\|U^{\overline{n}} - U^\infty\| \leqslant \frac{1}{\sqrt{c_h}}(|U^{\overline{n}} - U^\infty|_{-1,h} + |U^{\overline{n}} - U^\infty|_{0,\Gamma}) < \sigma/3$ and $E^h(U^{\overline{n}}) \leqslant \tilde{E}$. Let $N - 1 \geqslant \overline{n}$ be the largest integer (including $+\infty$) such that

$$\|U^n - U^\infty\| \leqslant \frac{1}{\sqrt{c_h}}(|U^n - U^\infty|_{-1,h} + |U^n - U^\infty|_{0,\Gamma}) < 2\sigma/3,$$

for all n with $\overline{n} \leqslant n \leqslant N - 1$. Assume by contradiction that N is finite. We deduce from (4.101) that

$$\begin{aligned}
\|U^N - U^\infty\| \leqslant & \frac{1}{\sqrt{c_h}}(|U^N - U^\infty|_{-1,h} + |U^N - U^\infty|_{0,\Gamma}) \\
\leqslant & \frac{1}{\sqrt{c_h}}\left(|U^N - U^{N-1}|_{-1,h} + |U^N - U^{N-1}|_{0,\Gamma}\right) \\
& + \frac{1}{\sqrt{c_h}}\left(|U^{N-1} - U^\infty|_{-1,h} + |U^{N-1} - U^\infty|_{0,\Gamma}\right) \\
\leqslant & \frac{1}{\sqrt{c_h}}\left(\sqrt{2\delta t E^h(U^{N-1})} + |U^{N-1} - U^\infty|_{-1,h} + |U^{N-1} - U^\infty|_{0,\Gamma}\right) \\
\leqslant & \sigma/3 + 2\sigma/3 = \sigma.
\end{aligned} \tag{4.110}$$

So we may apply (4.108) to every $\overline{n} \leqslant n \leqslant N - 1$ and since $(E^h(U^n))_n$ is non-increasing, we obtain

$$\sum_{n=\overline{n}}^{N} \|U^n - U^{n-1}\| \leqslant \frac{2^{2-\nu}\lambda_M \gamma}{\nu c_h} E^h(U^{N-1})^\nu + 5\left(\frac{\delta t}{c_h}\right)^{1/2} E^h(U^{N-1})^{1/2} \leqslant \sigma/3. \tag{4.111}$$

Thus,

$$\begin{aligned}
\|U^N - U^\infty\| & \leqslant \|U^N - U^{\overline{n}}\| + \|U^{\overline{n}} - U^\infty\| \\
& \leqslant \sum_{n=\overline{n}}^{N} \|U^n - U^{n-1}\| + \|U^{\overline{n}} - U^\infty\| \\
& \leqslant \sigma/3 + \sigma/3 = 2\sigma/3,
\end{aligned} \tag{4.112}$$

which is in contradiction with the definition of $N - 1$. So $N = +\infty$ and the whole sequence (U^n) converges to U_∞. Since w_h^n, defined by (4.86), is a continuous function of u_h^n, w_h^n also has a limit w_h^∞ as $n \to +\infty$. We see that (u_h^∞, w_h^∞) is necessarily a steady state by passing to the limit in (4.86). □

4.6 Numerical simulations

In this section, we illustrate some numerical simulations in two space dimensions. The fully discrete scheme (4.86) requires at each time step the resolution of a nonlinear system and for the numerical computation of solutions of the space semi-discrete scheme (4.14), we propose instead of (4.86) a semi-implicit time discretizations which

is the semi-implicit Euler (SIE) scheme, i.e. (4.14) but the implicit nonlinear terms $f(u_h^n)$ and $\tilde{g}(u_h^n)$ are respectively replaced by the explicit terms $f(u_h^{n-1})$ and $\tilde{g}(u_h^{n-1})$. Using the same arguments as in the proof of Proposition 4.3.1 we see that the matrix of the SIE scheme is positive semidefinite and invertible, so that the SIE scheme is well-posed.



Figure 4.1: $t = 5$

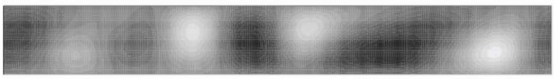

Figure 4.2: $t = 10$



Figure 4.3: $t = 25$



Figure 4.4: $t = 50$

In Figures 4.1-4.4, we see the result of the SIE scheme on the slab $L_x \times L_y = 80 \times 10$. The triangulation Ω^h was obtained by dividing the slab into 256×50 rectangles and by dividing every rectangle along the same diagonal into two triangles. The nonlinearities are

$$f(v) = \frac{1}{2}(v^3 - v) \quad \text{and} \quad \tilde{g}(v) = (\lambda + k)v - h, \quad v \in \mathbb{R}, \tag{4.113}$$

with $\lambda + k = 1$ ($\lambda = 0.5$, for instance), $h = 0$ and $\delta t = 0.1$. In each picture, the maximum and minimum values of u are colored in white and black and values of u in between correspond to different shades of grey. In these numerical simulations, we chose the same parameters as in [7] and [33]. Since $h = 0$, none of the components is preferably attracted by the walls, which is visible on the fact that both white and black zones appear at the boundary.



Figure 4.5: $t = 2$ $(h = 0)$



Figure 4.6: $t = 2$ $(h = 0.7)$

In Figures 4.5 and 4.6, we consider the nonlinearity $f(v) = v^3 - \frac{v}{2}, \quad v \in \mathbb{R}$, and the same nonlinearity $\tilde{g}(v) = (\lambda + k)v - h, v \in \mathbb{R}$. This time, $\delta t = 0.01$ and the geometry is different; the domain Ω is a disk of radius 80 centered at $(0, 0)$ from which we have cut off a disk of radius 40 and centered at $(20, 0)$. The exterior boundary is divided into 600 intervals and the internal boundary into 400 intervals, yielding a triangulation Ω^h of Ω with 59 048 triangles and 30 024 vertices. In these figures, we see the difference between the case $h = 0$, where no phase is preferentially attracted by the walls, and the case $h = 0.7$, where one of the components is preferentially attracted by the walls. We also remark that away from the boundary, Figures 4.5 and 4.6 present the same patterns.

Chapitre 5

Long Time Behavior of an Allen-Cahn Type Equation With a Singular Potential and Dynamic Boundary Conditions

Comportement asymptotique d'une équation de type Allen-Cahn avec un potentiel régulier et des conditions dynamiques sur le bord

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Long time behavior of an Allen-Cahn type equation with a singular potential and dynamic boundary conditions

Haydi ISRAEL

UMR 7348 CNRS. Laboratoire de Mathématiques et Applications - Université de Poitiers
SP2MI - Boulevard Marie et Pierre Curie - Téléport 2
BP30179 - 86962 Futuroscope Chasseneuil Cedex - FRANCE.

Abstract. The aim of this paper is to study the well-posedness and the long time behavior of solutions for an equation of Allen-Cahn type owing to proper approximations of the singular potential and a suitable definition of solutions. We also prove the existence of the finite dimensional global attractor as well as exponential attractors.

5.1 Introduction

In this article we are interested in the study of the following initial and boundary value problem, considered in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$:

$$\begin{cases} \partial_t \phi = \Delta\mu - \mu = -(-\Delta + I)\mu = -A\mu,\ \partial_n \mu_{|\partial\Omega} = 0, \\ \mu = -\Delta\phi + f(\phi) - \lambda\phi,\ \phi_{|t=0} = \phi_0, \\ \partial_t \psi = \Delta_\Gamma \psi - g(\psi) - \partial_n \phi,\ x \in \partial\Omega,\ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi, \end{cases} \tag{5.1}$$

where $\lambda \in \mathbb{R}$, Δ_Γ is the Laplace-Beltrami operator on the boundary $\partial\Omega$, f and g are given nonlinear interaction functions. In particular, f is the derivative of a double-well potential whose wells correspond to the phases of the material. A thermodynamically relevant function f is the following logarithmic (singular) function:

$$f(s) = -2\kappa_0 s + \kappa_1 \ln \frac{1+s}{1-s},\ s \in (-1,1),\ \kappa_0 > \kappa_1 > 0.$$

The boundary condition will be interpreted as an additional second-order parabolic equation on the boundary $\partial\Omega$.
Equation (5.1) may be viewed as a combination of the well-known Cahn-Hilliard equation

$$\partial_t u = -\Delta(\Delta u + f(u)), \quad u(0,x) = u_0(x),$$

and of the Allen-Cahn equation

$$\partial_t u = \Delta u + f(u), \quad u(0,x) = u_0(x).$$

This equation is associated with multiple microscopic mechanisms such as surface diffusion and adsorption/desorption and was recently derived and studied in Karali & Katsoulakis [30], Katsoulakis & Vlachos [32], Israel [28], Mikhailov, Hildebrand & Ertl [37].
This paper is organized as follows. In Section 2, we introduce regularized problems in which the singular nonlinearity is approximated by regular functions and we derive uniform a priori estimates on the corresponding solutions. In Section 3, we formulate the variational formulation of (5.2), we verify the existence and uniqueness of a solution and we study the further regularity of the solutions. In Section 4, we give sufficient conditions which ensure that solutions are separated from the singularities of f and that a variational solution coincides with a solution in the usual (distribution) sense. Finally, we study in Section 5 the asymptotic behavior of the system and we prove the existence of finite-dimensional (both global and exponential) attractors.

5.2 Approximations and uniform a priori estimates

We set $\tilde{f}(\phi) := f(\phi) - \lambda\phi$ and rewrite problem (5.1) in the form:

$$\begin{cases} \partial_t \phi = \Delta\mu - \mu = -(-\Delta + I)\mu = -A\mu, \ \partial_n \mu_{|\partial\Omega} = 0, \\ \mu = -\Delta\phi + \tilde{f}(\phi), \ \phi_{|t=0} = \phi_0, \\ \partial_t \psi = \Delta_\Gamma \psi - g(\psi) - \partial_n \phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi, \end{cases} \tag{5.2}$$

where the singular function f satisfies:

$$\begin{cases} f \in C^2((-1,1)), \\ f(0) = 0, \ \lim\limits_{s\to\pm 1} f(s) = \pm\infty, \\ f'(s) \geqslant 0, \ \lim\limits_{s\to\pm 1} f'(s) = +\infty, \\ f''(s)\ \mathrm{sgn}\ s \geqslant 0. \end{cases} \tag{5.3}$$

As a consequence, the following properties hold for $\tilde{f}$:

$$\tilde{f}'(s) \geqslant -\lambda \text{ and } -\tilde{c} \leqslant \tilde{F}(s) \leqslant \tilde{f}(s)s + \tilde{C}, \ \ \forall s \in (-1,1), \tag{5.4}$$

where $\tilde{F}(s) = \int_0^s \tilde{f}(r)dr$ and $\tilde{c}$, $\tilde{C}$ are strictly positive constants.
The nonlinear function $g \in C^2([-1,1])$ can be extended, without loss of generality, to the whole real line by writing:

$$g(s) = s + g_0(s), \quad \forall s \in \mathbb{R}, \text{ where } \|g_0\|_{C^2(\mathbb{R})} := C_0 < +\infty. \tag{5.5}$$

We set, for $r \geqslant 1$,

$$H^r(\Omega) \otimes H^r(\Gamma) := \left\{ v \in H^r(\Omega), \ v|_\Gamma \in H^r(\Gamma) \right\},$$

which we endow with the norm:

$$\|v\|^2_{H^r(\Omega)\otimes H^r(\Gamma)} = \|v\|^2_{H^r(\Omega)} + \|v\|^2_{H^r(\Gamma)}.$$

Alternatively, the functions in $H^r(\Omega) \otimes H^r(\Gamma)$ can be viewed as pairs of functions $(v, v|_\Gamma)$.
We introduce a family of regular approximating functions: given any $N \in \mathbb{N}$, we set:

$$f_N(s) = \begin{cases} f(-1+1/N) + f'(-1+1/N)(s+1-1/N), & s < -1+1/N, \\ f(s), \quad |s| \leqslant 1-1/N, \\ f(1-1/N) + f'(1-1/N)(s-1+1/N), & 1-1/N < s. \end{cases} \tag{5.6}$$

Then, we denote F_N the primitive $F_N(s) = \int_0^s f_N(s)\mathrm{d}s$ and having set $\tilde{f}_N(s) = f_N(s) - \lambda s$, we define $\tilde{F}_N$ analogously, with $\tilde{f}_N$ instead of f_N. We recall the following properties (for more details, see Miranville & Zelik [45]), namely, there exist $\alpha > 0$, $c > 0$ and $C > 0$ such that:

$$\tilde{f}_N(s)s \geqslant \alpha/2|f_N(s)| - c, \tag{5.7}$$

and

$$1/2F_N(s) - C \leqslant \tilde{F}_N(s) \leqslant 2F_N(s) + C, \tag{5.8}$$

$\forall s \in \mathbb{R}$ and for $N \geqslant N_0(\lambda)$ large enough, where the constant C only depends on λ.
We then consider the approximate problems:

$$\begin{cases} \partial_t \phi = \Delta\mu - \mu = -(-\Delta + I)\mu = -A\mu, \ \partial_n \mu_{|\partial\Omega} = 0, \\ \mu = -\Delta\phi + \tilde{f}_N(\phi), \ \phi_{|t=0} = \phi_0, \\ \partial_t \psi = \Delta_\Gamma \psi - g(\psi) - \partial_n \phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi. \end{cases} \tag{5.9}$$

It is convenient to rewrite problem (5.9) in an equivalent form by using the inverse of $A := (-\Delta + I)$ (endowed with Neumann boundary conditions). Applying A^{-1} to both side of (5.9), we obtain:

$$\begin{cases} A^{-1}\partial_t \phi - \Delta\phi + \tilde{f}_N(\phi) = 0, \ x \in \Omega, \ \phi_{|t=0} = \phi_0, \\ \partial_t \psi = \Delta_\Gamma \psi - g(\psi) - \partial_n \phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi. \end{cases} \tag{5.10}$$

We start with the usual energy equality.

Lemma 5.2.1. *Let the above assumptions hold and let ϕ be a sufficiently regular solution of* (5.10)*. Then, the following identities hold:*

$$\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} + \int_0^t \left(\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} + (\tilde{F}_N(\phi(s)), 1)_\Omega \right) \mathrm{d}s \leqslant C \left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)} \right), \tag{5.11}$$

$$\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} + \int_0^t e^{-\nu(t-s)} \left(\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} \right) \mathrm{d}s + \int_0^t e^{-\nu(t-s)} \|f_N(\phi(s)\|_{L^1(\Omega)} \mathrm{d}s \leqslant C \left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)} \right) e^{-\nu t}, \tag{5.12}$$

and

$$\int_t^{t+1} (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} + \alpha \|f_N(\phi(s)\|_{L^1(\Omega)}) \mathrm{d}s \leqslant C \left(1 + (\|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}) e^{-\nu t} \right), \tag{5.13}$$

for some positive constants C and ν which are independent of t.

Proof:
Multiplying (5.10) by ϕ and using the fact that:

$$\|\phi\|_{H^1(\Omega)} \leqslant C(\|\nabla \phi\|_{L^2(\Omega)} + \|\psi\|_{H^1(\Gamma)}), \tag{5.14}$$

we find:

$$\frac{1}{2} \frac{d}{dt} \left(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} \right) + \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + (\tilde{f}_N(\phi(t)), \phi(t))_\Omega + (g_0(\psi(t)), \psi(t))_\Gamma \leqslant 0. \tag{5.15}$$

Using (5.4), the fact that g_0 is globally bounded and that $\|\psi(t)\|_{L^\infty(\Gamma)} \leqslant 1$, we obtain:

$$\frac{d}{dt} \left(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} \right) + \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + (\tilde{F}_N(\phi(t)), 1)_\Omega \leqslant C, \tag{5.16}$$

for some positive constant C. Integrating (5.16) with respect to t, we deduce:

$$\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} + \int_0^t \left(\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} + (\tilde{F}_N(\phi(s)), 1)_\Omega \right) \mathrm{d}s \leqslant C \left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)} \right),$$

for some constant C.
Now, using (5.7) and the fact that g_0 is globally bounded, (5.15) gives:

$$\frac{1}{2} \frac{d}{dt} \left(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} \right) + \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + \alpha/2 \|f_N(\phi(t)\|_{L^1(\Omega)} \leqslant C, \tag{5.17}$$

for some positive constants α and C. Hence, for $\nu > 0$ small enough, we obtain:

$$\frac{d}{dt} (\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}) + \nu \left(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} \right) + c_\nu (\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + \|f_N(\phi(t)\|_{L^1(\Omega)}) \leqslant C. \tag{5.18}$$

Applying Gronwall's lemma, we deduce estimate (5.12). Finally, integrating (5.17) with respect to t over $(t, t+1)$ and using (5.12), we obtain:

$$\begin{aligned}\int_t^{t+1} (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} + \alpha\|f_N(\phi(s)\|_{L^1(\Omega)})\mathrm{d}s \\ \leqslant C(1 + \|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}) \\ \leqslant C\left(1 + (\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)})e^{-\nu t}\right).\end{aligned} \tag{5.19}$$

□

Lemma 5.2.2. *Let the assumptions of Lemma 5.2.1 hold. Then, the following identity holds:*

$$\begin{aligned}&\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)), 1)_\Omega \\ &\qquad + \int_0^t (\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ &\leqslant C\left(1 + \|\phi(0)\|^2_{H^1(\Omega)} + \|\psi(0)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(0)), 1)_\Omega\right),\end{aligned} \tag{5.20}$$

where the constant C is independent of t and of the initial data.

Proof:
Multiplying the first equation of (5.10) by $\partial_t\phi$ and integrating over Ω, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + \|\psi(t)\|^2_{L^2(\Gamma)}\right) + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} \\ &\quad + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)} + (\tilde{f}_N(\phi(t)), \partial_t\phi(t))_\Omega + (g_0(\psi(t)), \partial_t\psi(t))_\Gamma \\ =&\frac{1}{2}\frac{d}{dt}(\|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + \|\psi(t)\|^2_{L^2(\Gamma)} + 2(\tilde{F}_N(\phi(t)), 1)_\Omega \\ &\quad + 2(G_0(\psi(t)), 1)_\Gamma) + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)} \\ =&0,\end{aligned} \tag{5.21}$$

where $G_0(t) = \int_0^t g_0(s)\mathrm{d}s$.
Using (5.14), we find:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)), 1)_\Omega + 2(G_0(\psi(t)), 1)_\Gamma\right) \\ &+\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)} = 0.\end{aligned} \tag{5.22}$$

Integrating (5.22) with respect of t, taking into account that g_0 is globally bounded, we deduce (5.20). □

Lemma 5.2.3. *Let the assumptions of Lemma 5.2.1 hold, ϕ be a sufficiently regular solution of* (5.10) *and N be large enough. Then, for $t > 0$, the following smoothing property holds:*

$$\begin{aligned} & t\left(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega\right) \\ & \qquad + \int_0^t s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ & \leqslant C\left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}\right), \end{aligned} \tag{5.23}$$

where the constant C is independent of N.

Proof:
Multiplying (5.22) by t, we obtain:

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(t(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega + 2(G_0(\psi(t)),1)_\Gamma)) \\ & \qquad + t(\|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)}) \\ = & \frac{1}{2}(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega + 2(G_0(\psi(t)),1)_\Gamma). \end{aligned} \tag{5.24}$$

Integrating (5.24) with respect of t from 0 to t, we have:

$$\begin{aligned} & t(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega + 2(G_0(\psi(t)),1)_\Gamma \\ & \quad + 2\int_0^t s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ & \leqslant \int_0^t (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(s)),1)_\Omega + 2(G_0(\psi(s)),1)_\Gamma). \end{aligned} \tag{5.25}$$

Using (5.8), (5.11) and that g_0 is bounded globally, we deduce:

$$\begin{aligned} & t\left(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega\right) \\ & \qquad + \int_0^t s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ & \leqslant C\left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}\right), \end{aligned} \tag{5.26}$$

and the proof is complete. □

Lemma 5.2.4. *Let the assumptions of Lemma 5.2.1 hold. Then, we have, for all $t \geqslant 1$ and N large enough, the following property:*

$$\begin{aligned} & \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega \\ & \qquad + \int_1^t (\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ & \leqslant C(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}). \end{aligned} \tag{5.27}$$

Moreover, for any $t > 0$ and N large enough, the following inequality holds:

$$\begin{aligned} & \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega \\ & \qquad \leqslant C\frac{t+1}{t}(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}). \end{aligned} \tag{5.28}$$

Proof:
Integrating (5.22) with respect to t from 1 to t, we obtain:

$$\begin{aligned}&\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega \\ &\qquad + \int_1^t (\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ &\leqslant C\left(1 + \|\phi(1)\|^2_{H^1(\Omega)} + \|\psi(1)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(1)),1)_\Omega\right).\end{aligned} \tag{5.29}$$

From (5.23) and for $t = 1$, we find:

$$\begin{aligned}&\left(\|\phi(1)\|^2_{H^1(\Omega)} + \|\psi(1)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(1)),1)_\Omega\right) \\ &\qquad + \int_0^1 s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ &\leqslant C\left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}\right).\end{aligned} \tag{5.30}$$

Estimates (5.29) and (5.30) allow us to find (5.27). Estimate (5.28) follows immediately from (5.23) and (5.27). □
We will now give some additional regularity results on $\partial_t\phi(t)$. To prove this, we differentiate (5.9) and set $(u(t), v(t), w(t)) := \partial_t(\phi(t), \mu(t), \psi(t))$. Then we have:

$$\begin{cases} \partial_t u = \Delta v - v = -(-\Delta + I)v = -Av,\ \partial_n v_{|\Gamma} = 0, \\ v = -\Delta u + \tilde{f}'_N(\phi)u, \\ \partial_t w = \Delta_\Gamma w - g'(\psi)w - \partial_n w,\ x \in \Gamma, \\ u_{|\Gamma} = w. \end{cases} \tag{5.31}$$

Lemma 5.2.5. *Let the assumptions of Lemma 5.2.1 hold. Then, the following estimate is valid for all $t > 0$:*

$$\begin{aligned}\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} + \int_t^{t+1} (\|u(s)\|^2_{H^1(\Omega)} + \|w(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ \leqslant c(\|u(0)\|^2_{H^{-1}(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)})e^{-\nu t} + c,\end{aligned} \tag{5.32}$$

for some positive constants c and ν independent of N. Moreover, for $t > 0$, we have the smoothing property:

$$\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} \leqslant c\frac{t^2+1}{t^2}(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}). \tag{5.33}$$

Proof:
Multiplying the first equation of (5.31) by $A^{-1}u$, the second equation by u and the third one by w and taking the sum of the equations that we obtain, we have the following identity:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}\right) + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma w(t)\|^2_{L^2(\Gamma)} + \|w(t)\|^2_{L^2(\Gamma)} \\ + (\tilde{f}'_N(\phi(t))u(t), u(t))_\Omega + (g'_0(\psi(t))w(t), w(t))_\Gamma = 0.\end{aligned} \tag{5.34}$$

Using (5.4), (5.14) and that g_0' is globally bounded, we find:

$$\begin{aligned}\frac{d}{dt}\left(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}\right)+&2\|u(t)\|^2_{H^1(\Omega)}+2\|w(t)\|^2_{H^1(\Gamma)}\\ &\leqslant c(\|u(t)\|^2_{L^2(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}).\end{aligned}\tag{5.35}$$

Using the interpolation inequality $\|u\|^2_{L^2(\Omega)}\leqslant c\|u\|_{H^1(\Omega)}\|u\|_{H^{-1}(\Omega)}$, we find:

$$\begin{aligned}\frac{d}{dt}\left(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}\right)+&\|u(t)\|^2_{H^1(\Omega)}+\|w(t)\|^2_{H^1(\Gamma)}\\ &\leqslant c(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}).\end{aligned}\tag{5.36}$$

Provided that $\nu>0$ is small enough, the following estimate holds:

$$\begin{aligned}\frac{d}{dt}(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)})+\nu(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)})\\ +c_\nu(\|u(t)\|^2_{H^1(\Omega)}+\|w(t)\|^2_{H^1(\Gamma)})\leqslant c(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}).\end{aligned}\tag{5.37}$$

We consider the case $t\geqslant 1$. Applying Gronwall's inequality and using estimate (5.27), we obtain:

$$\begin{aligned}&\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}\leqslant(\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)})e^{-\nu(t-1)}\\ &\qquad+c\int_1^t e^{-\nu(t-s)}(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ &\leqslant(\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)})e^{-\nu(t-1)}+c\int_1^t(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ &\leqslant(\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)})e^{-\nu(t-1)}+c(1+\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)}).\end{aligned}\tag{5.38}$$

Integrating (5.37) over $(t,t+1),\ t\geqslant 1$, we find:

$$\begin{aligned}&\int_t^{t+1}(\|u(s)\|^2_{H^1(\Omega)}+\|w(s)\|^2_{H^1(\Gamma)})\mathrm{d}s\\ \leqslant&\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}+c\int_t^{t+1}(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ \leqslant&(\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)})e^{-\nu(t-1)}+c(1+\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)}).\end{aligned}\tag{5.39}$$

We have:

$$\begin{aligned}t^2(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)})=&2\int_0^t s(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ &+\int_0^t s^2\partial_t(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s.\end{aligned}$$

Taking into account (5.26) and (5.36), we obtain for $t \in (0,1]$:

$$\begin{aligned} t^2(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}) \leqslant & 2\int_0^t s(\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ & + c\int_0^t s(\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ \leqslant & (2+c)\int_0^t s(\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ \leqslant & c(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}), \end{aligned} \tag{5.40}$$

for some positive constant c. For $t = 1$, estimate (5.40) gives:

$$\|u(1)\|^2_{H^{-1}(\Omega)} + \|w(1)\|^2_{L^2(\Gamma)} \leqslant c(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}), \tag{5.41}$$

then, replacing this estimate in (5.38) and (5.39), we deduce:

$$\begin{aligned} \|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} + & \int_t^{t+1} (\|u(s)\|^2_{H^1(\Omega)} + \|w(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ & \leqslant c + c(\|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)})e^{-\nu t}, \end{aligned} \tag{5.42}$$

for all $t \geqslant 1$. From estimates (5.40) and (5.42), we deduce:

$$\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} \leqslant c(1 + \frac{1}{t^2})(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}),\ \forall t > 0. \tag{5.43}$$

To show estimate (5.32) for $t \in (0,1]$, we consider (5.37), in particular,

$$\frac{d}{dt}(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}) \leqslant c(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}). \tag{5.44}$$

Applying Gronwall's inequality to (5.44), we find:

$$\begin{aligned} \|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} \leqslant & ce^{ct}(\|u(0)\|^2_{H^{-1}(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)}) \\ \leqslant & c'e^{-\nu t}(\|u(0)\|^2_{H^{-1}(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)}),\ t \in (0,1]. \end{aligned} \tag{5.45}$$

Integrating (5.37) over $(t, t+1)$, $t \in (0,1]$, using (5.42) and (5.45), we obtain:

$$\begin{aligned} & \int_t^{t+1} (\|u(s)\|^2_{H^1(\Omega)} + \|w(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ \leqslant & \|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} + c\int_t^{t+1} (\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ \leqslant & \|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} + c\int_t^{1} (\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ & + c\int_1^{t+1} (\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ \leqslant & c + c(\|u(0)\|^2_{H^{-1}(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)})e^{-\nu t}. \end{aligned} \tag{5.46}$$

Hence, the proof is complete. □

Theorem 5.2.6. *Let the nonlinearities f and g satisfy* (5.3) *and* (5.5) *respectively and set $\Omega_\delta := \{x \in \Omega,\ d(x,\Gamma) > \delta\}$. Denote by $n = n(x)$ some smooth extension of the unit normal vector field at the boundary inside the domain Ω. Let also $D_\tau\phi := \nabla_x\phi - (\partial_n\phi)_n$ be the tangential part of the gradient $\nabla_x\phi$. Then, for every $\delta > 0$, the following estimate is valid:*

$$\begin{aligned}\|\phi(t)\|^2_{C^\alpha(\Omega)} + \|\phi(t)\|^2_{H^2(\Omega_\delta)} + \|\psi(t)\|^2_{H^2(\Gamma)} + \|\phi(t)\|^2_{H^1(\Omega)} + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)}\\ +\|\partial_t\psi(t)\|^2_{L^2(\Gamma)} + \|\nabla D_\tau\phi(t)\|^2_{[L^2(\Omega)]^6} + \|f_N(\phi(t))\|_{L^1(\Omega)}\\ + \int_t^{t+1}(\|\partial_t\phi(s)\|^2_{H^1(\Omega)} + \|\partial_t\psi(s)\|^2_{H^1(\Gamma)})\mathrm{d}s\end{aligned}$$

$$\leqslant C(1 + \|\phi(0)\|^2_{H^1(\Omega)} + \|\psi(0)\|^2_{H^1(\Gamma)} + \|\partial_t\phi(0)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(0)\|^2_{L^2(\Gamma)})e^{-\gamma t}, \quad (5.47)$$

where the positive constants $\alpha(\alpha > 1/4)$, γ and C are independent of N.

Proof:
We consider the nonlinear elliptic problem:

$$\begin{cases} \Delta\phi(t) - f_N(\phi(t)) - \phi(t) = h_1(t) := -\phi(t) + \lambda\phi(t) + A^{-1}\partial_t\phi(t), & x \in \Omega \\ \Delta_\Gamma\psi(t) - \psi(t) - \partial_n\phi(t) = h_2(t) := g_0(\psi(t)) + \partial_t\psi(t), & x \in \Gamma, \end{cases} \quad (5.48)$$

for every fixed t. Note here that the estimates derived above yield the following control of the right-hand side of (5.48):

$$\|h_1(t)\|^2_{L^2(\Omega)} + \|h_2(t)\|^2_{L^2(\Gamma)} \leqslant C(1 + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)}), \quad (5.49)$$

where C is a positive constant that is independent of N. Due to estimate (5.33), we find that $h_1 \in L^2(\Omega)$ and $h_2 \in L^2(\Gamma)$. Using estimates (5.20), (5.32) and (5.80), we obtain:

$$\begin{aligned}&\|\phi(t)\|^2_{H^1(\Omega)} + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)} + \|f_N(\phi(t))\|_{L^1(\Omega)}\\ &\qquad + \int_t^{t+1}(\|\partial_t\phi(s)\|^2_{H^1(\Omega)} + \|\partial_t\psi(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \quad (5.50)\\ &\leqslant C(1 + \|\phi(0)\|^2_{H^1(\Omega)} + \|\psi(0)\|^2_{H^1(\Gamma)} + \|\partial_t\phi(0)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(0)\|^2_{L^2(\Gamma)})e^{-\gamma t}.\end{aligned}$$

In order to prove the following estimate:

$$\|\phi(t)\|^2_{H^2(\Omega_\delta)} \leqslant C(1 + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)}), \quad (5.51)$$

where $C = C_\epsilon$ depends on $\epsilon > 0$. We consider a smooth nonnegative cut-off function θ such that $\theta(x) = 1$ if $d(x,\Gamma) \geqslant \delta$ and $\theta(x) = 0$ if $d(x,\Gamma) \leqslant \delta/2$ which satisfies, in addition, the inequality:

$$|\nabla_x\theta(x)| \leqslant C\theta^{1/2}(x).$$

Then, we multiply equation (5.48) by $\sum_{i=3}^3 \partial_{x_1}(\theta(x)\partial_{x_i}u)$, and we integrate by parts. Using estimate (5.50) and the fact that $f' \geqslant 0$, we obtain estimate (5.51). In order to prove:

$$\|\nabla_x D_\tau\phi\|^2_{L^2(\Omega)^6} + \|\phi\|^2_{H^2(\Gamma)} \leqslant C(1 + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)}), \quad (5.52)$$

we study the function ϕ in a small ϵ-neighborhood of the boundary Γ. To do so, let $x_0 \in \Gamma$ and $y = y(x)$ be a local coordinates in the neighborhood of x_0 such that $y(x_0) = 0$ and Ω is defined, in these coordinates, by the condition $y_1 > 0$. Then, we rewrite problem (5.48) in the variable y and after several transformations we find estimate (5.52). To finish the proof of the theorem, we use the following embedding:

$$L^2(\mathbb{R}, H^2(\mathbb{R}^2)) \cap H^1(\mathbb{R}, H^1(\mathbb{R}^2)) \subset C^\alpha(\mathbb{R}^3),\ \alpha < 1/4,$$

and we deduce the estimate:

$$\|\phi\|^2_{C^\alpha(\Omega)} \leqslant C(1 + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)}).$$

Hence, Theorem (5.2.6) is proved (for more details see Miranville & Zelik [45]). □
In what follows, we will establish the uniform Lipschitz continuity of the solution $(\phi(t), \mu(t), \psi(t))$ of problem (5.10) with respect to the initial data.

Proposition 5.2.7. *Let the above assumptions hold and let $(\phi_1(t), \mu_1(t), \psi_1(t))$ and $(\phi_2(t), \mu_2(t), \psi_2(t))$ be two solutions of problem (5.10). Then, the following estimate holds:*

$$\begin{aligned} &\|\phi_1(t) - \phi_2(t)\|^2_{H^{-1}(\Omega)} + \|\psi_1(t) - \psi_2(t)\|^2_{L^2(\Gamma)} \\ &\qquad + \int_t^{t+1} (\|\phi_1(s) - \phi_2(s)\|^2_{H^1(\Omega)} + \|\psi_1(s) - \psi_2(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ &\leqslant C\left(\|\phi_1(0) - \phi_2(0)\|^2_{H^{-1}(\Omega)} + \|\psi_1(0) - \psi_2(0)\|^2_{L^2(\Gamma)}\right) e^{Kt}, \end{aligned} \tag{5.53}$$

where the constants C and K are independent of t, N and the initial data.

Proof:
Let $(\phi(t), \mu(t), \psi(t)) = (\phi_1(t) - \phi_2(t), \mu_1(t) - \mu_2(t), \psi(t) - \psi_2(t))$. Then, this function satisfies the system:

$$\begin{cases} \partial_t \phi = -A\mu,\ \partial_n \mu_{|\partial\Omega} = 0, \\ \mu = -\Delta\phi + \tilde{l}_N(t)\phi,\ \phi_{|t=0} = \phi_0, \\ \partial_t \psi = \Delta_\Gamma \psi - \psi - m(t)\psi - \partial_n \phi,\ x \in \partial\Omega,\ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi, \end{cases} \tag{5.54}$$

where

$$\tilde{l}_N(t) := \int_0^t \tilde{f}'_N(s\phi_1(t) + (1-s)\phi_2(t))\mathrm{d}s \text{ and } m(t) := \int_0^t g'_0(s\phi_1(t) + (1-s)\phi_2(t))\mathrm{d}s.$$

Multiplying the first equation of (5.31) by $A^{-1}\phi$, the second equation by ϕ and the third one by ψ and taking the sum of the equations that we obtain, we have the following identity:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}\right) + \|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + \|\psi(t)\|^2_{L^2(\Gamma)} \\ &+ (\tilde{l}_N(t)\phi(t), \phi(t))_\Omega + (m(t)\psi, \psi(t))_\Gamma = 0. \end{aligned} \tag{5.55}$$

Using (5.4), (5.14) and the fact that g_0' is globally bounded, we obtain:

$$\begin{aligned}\frac{d}{dt}\left(\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)}\right)+\alpha'\left(\|\phi(t)\|^2_{H^1(\Omega)}+\|\psi(t)\|^2_{H^1(\Omega)}\right)\\ \leqslant C(\|\phi(t)\|^2_{L^2(\Omega)}+\|\psi(t)\|^2_{L^2(\Omega)})\end{aligned} \tag{5.56}$$

for some positive constants α' and C which are independent of N. Using the interpolation inequality $\|u\|_2^2 \leqslant C\|u\|_{H^1(\Omega)}\|u\|_{H^{-1}(\Omega)}$ and applying the Gronwall inequality, we deduce (5.53). □

5.3 Variational formulation and well-posedness

This section is devoted to the definition of a suitable notion for a solution to the limit problem, that is, the problem obtained by letting $N \to +\infty$ and which coincides with (5.2).To this end, we first fix a constant $L > 0$ such that:

$$\|\nabla\varphi\|^2_{L^2(\Omega)}-\lambda\|\varphi\|^2_{L^2(\Omega)}+L\|\varphi\|^2_{H^{-1}(\Omega)} \geqslant 1/2\|\varphi\|^2_{H^1(\Omega)}, \tag{5.57}$$

for all $\varphi \in H^1(\Omega)$ and introduce the quadratic form:

$$B(\varphi,\rho) := (\nabla\varphi,\nabla\rho)_\Omega-\lambda(\varphi,\rho)_\Omega+L((-\Delta+I)^{-1}\varphi,\rho)_\Omega+(\nabla_\Gamma\varphi,\nabla_\Gamma\rho)_\Gamma,$$

$\forall\varphi,\rho \in H^1(\Omega)\otimes H^1(\Gamma)$. Then, obviously, we have:

$$B(\varphi,\varphi) \geqslant 0, \quad \forall\varphi \in H^1(\Omega)\otimes H^1(\Gamma).$$

The limit problem (5.10), corresponding to $N = +\infty$ formally reads:

$$\begin{cases} A^{-1}\partial_t\phi = \Delta\phi - f(\phi)+\lambda\phi,\ \phi_{|t=0}=\phi_0,\\ \partial_t\psi = \Delta_\Gamma\psi - g(\psi)-\partial_n\phi,\ x\in\partial\Omega,\ \psi_{|t=0}=\psi_0,\\ \phi_{|\partial\Omega}=\psi. \end{cases} \tag{5.58}$$

Multiplying the first equation of (5.58) by the function $\phi-\varphi$, where $\varphi=\varphi(t,x)$ is smooth, and integrating by parts, we obtain:

$$\begin{aligned}(A^{-1}\partial_t\phi,\phi-\varphi)_\Omega+(\partial_t\phi,\phi-\varphi)_\Gamma+(\nabla\phi,\nabla(\phi-\varphi))_\Omega-\lambda(\phi,\phi-\varphi)_\Omega+(\nabla_\Gamma\phi,\nabla_\Gamma(\phi-\varphi))_\Gamma\\ +(f(\phi),\phi-\varphi)_\Omega+(g(\phi),\phi-\varphi)_\Gamma=0,\end{aligned}$$

which yields:

$$\begin{aligned}(A^{-1}\partial_t\phi,\phi-\varphi)_\Omega+(\partial_t\phi,\phi-\varphi)_\Gamma+B(\phi,\phi-\varphi)_\Omega+(f(\phi),\phi-\varphi)_\Omega\\ =L(A^{-1}\phi,\phi-\varphi)_\Omega-(g(\phi),\phi-\varphi)_\Gamma,\end{aligned} \tag{5.59}$$

$\forall\varphi \in H^1(\Omega)\otimes H^1(\Gamma)$. Finally, since B is positive and f is monotone, we have:

$$B(\phi,\phi-\varphi) \geqslant B(\varphi,\phi-\varphi), \quad (f(\phi),\phi-\varphi)_\Omega \geqslant (f(\varphi),\phi-\varphi)_\Omega. \tag{5.60}$$

Consequently, (5.59) can be written as follow:

$$\begin{aligned}(A^{-1}\partial_t\phi,\phi-\varphi)_\Omega+(\partial_t\phi,\phi-\varphi)_\Gamma+B(\varphi,\phi-\varphi)_\Omega+(f(\varphi),\phi-\varphi)_\Omega\\ \leqslant L(A^{-1}\phi,\phi-\varphi)_\Omega-(g(\phi),\phi-\varphi)_\Gamma,\end{aligned} \tag{5.61}$$

$\forall \varphi \in H^1(\Omega) \otimes H^1(\Gamma)$. If we consider the solutions of problem (5.10) with initial data belonging to:

$$\Phi := \{(\phi, \psi) \in L^\infty(\Omega) \times L^\infty(\Gamma),\ \|\phi\|_{L^\infty(\Omega)} \leqslant 1,\ \|\psi\|_{L^\infty(\Gamma)} \leqslant 1\}$$

and then pass to the limit $N \to \infty$, we will find functions living in Φ for all time. These functions are not necessarily solutions to (5.2) in the usual sense. For this, we define a variational solution of the limit problem (5.58) as follows.

Definition 5.3.1. Let $(\phi_0, \psi_0) \in \Phi$. We say that $(\phi(t), \psi(t))$ is a variational solution to problem (5.2) originating from (ϕ_0, ψ_0) if

1. $\phi(t)|_\Gamma = \psi(t)$ for almost all $t > 0$,
2. $\phi(0) = \phi_0,\ \psi(0) = \psi_0$,
3. $-1 < \phi(t, x) < 1$ for almost all $(t, x) \in \mathbb{R}^+ \times \Omega$,
4. $(\phi, \psi) \in C([0, +\infty), H^{-1}(\Omega) \times L^2(\Gamma)) \cap L^2([0, T], H^1(\Omega) \times H^1(\Gamma))$, for any $T > 0$,
5. $f(\phi) \in L^1([0, T] \times \Omega)$ for any $T > 0$,
6. $(\partial_t \phi, \partial_t \psi) \in L^2([0, T], H^{-1}(\Omega) \times L^2(\Gamma))$, for any $T > 0$,

and the variational inequality

$$\begin{aligned}(A^{-1}\partial_t\phi(t), \phi(t) - \varphi)_\Omega + (\partial_t\phi(t), \phi(t) - \varphi)_\Gamma + B(\varphi, \phi(t) - \varphi)_\Omega + (f(\varphi), \phi(t) - \varphi)_\Omega \\ \leqslant L(A^{-1}\phi(t), \phi(t) - \varphi)_\Omega - (g(\phi(t)), \phi(t) - \varphi)_\Gamma,\end{aligned} \tag{5.62}$$

is satisfied for almost all $t > 0$ and any test function $\varphi \in H^1(\Omega) \otimes H^1(\Gamma)$ such that $f(\varphi) \in L^1(\Omega)$.

We emphasize that we do not assume in the definition that ψ_0 is the trace of ϕ_0. In order to show the uniqueness of a variational solution, we consider (5.62) in terms of test functions $\varphi = \varphi(t, x)$ depending on t and x with φ satisfying the regularity assumptions in Definition 5.3.1. Then, we write inequality (5.62) with $\varphi = \varphi(t, x)$ for almost all $t > 0$. Moreover, due to the regularity assumptions (5.3.1) on ϕ and φ, we integrate (5.62) with respect to t since all terms are in L^1. This gives, for all $t > s > 0$:

$$\begin{aligned}&\int_s^t \big((A^{-1}\partial_t\phi, \phi - \varphi)_\Omega + (\partial_t\phi, \phi - \varphi)_\Gamma + B(\varphi, \phi - \varphi)_\Omega + \big(f(\varphi), \phi - \varphi\big)_\Omega\big)\, \mathrm{d}\tau \\ \leqslant &\int_s^t \big(L(\phi, A^{-1}(\phi - \varphi))_\Omega - (g(\phi), \phi - \varphi)_\Gamma\big)\, \mathrm{d}\tau.\end{aligned} \tag{5.63}$$

Arguing as in Miranville & Zelik [45], we set $\varphi_\alpha := (1 - \alpha)\phi + \alpha\varphi$, where $\alpha \in (0, 1]$. Then, assumption $(5.3)_4$ implies that the function $|f(\phi)|$ is convex and

$$|f(\varphi_\alpha)| \leqslant |f(\phi)| + |f(\varphi)|,$$

which yields that $f(\varphi_\alpha) \in L^1(\Omega)$. Consequently, φ_α is an admissible test function for (5.63). Inserting $\varphi = \varphi_\alpha$ in the variational inequality (5.63), simplifying by α and using the fact that (ϕ, ψ) is absolutely continuous on $[s,t]$ with values in $H^{-1}(\Omega) \times L^2(\Omega)$, we get:

$$\begin{aligned}\int_s^t ((A^{-1}\partial_t\phi, \phi-\varphi)_\Omega + (\partial_t\phi, \phi-\varphi)_\Gamma + B(\varphi_\alpha, \phi-\varphi)_\Omega + (f(\varphi_\alpha), \phi-\varphi)_\Omega)\mathrm{d}\tau \\ \leqslant \int_s^t \left(L(\phi, A^{-1}(\phi-\varphi))_\Omega - (g(\phi), \phi-\varphi)_\Gamma\right)\mathrm{d}\tau.\end{aligned} \tag{5.64}$$

Passing to the limit in (5.64) as $\alpha \to 0$ and using the Lebesgue dominated convergence theorem for the nonlinear term, we obtain:

$$\begin{aligned}\int_s^t ((A^{-1}\partial_t\phi, \phi-\varphi)_\Omega + (\partial_t\phi, \phi-\varphi)_\Gamma + B(\phi, \phi-\varphi)_\Omega + (f(\phi), \phi-\varphi)_\Omega)\mathrm{d}\tau \\ \leqslant \int_s^t \left(L(\phi, A^{-1}(\phi-\varphi))_\Omega - (g(\phi), \phi-\varphi)_\Gamma\right)\mathrm{d}\tau.\end{aligned} \tag{5.65}$$

We can now state the following theorem which gives the uniqueness of such variational solutions.

Theorem 5.3.2. *Let the nonlinearity f and g satisfy the assumptions of Section 1. Then, the variational solution of problem* (5.58) *(in the sense of Definition 5.3.1) is unique and is independent of the choice of L satisfying* (5.57)*. Furthermore, for every two variational solutions (ϕ_1, ψ_1) and (ϕ_2, ψ_2), we have the following estimate:*

$$\begin{aligned}\|\phi_1(t) - \phi_2(t)\|^2_{H^{-1}(\Omega)} + \|\psi_1(t) - \psi_2(t)\|^2_{L^2(\Gamma)} \\ \leqslant ce^{Kt}(\|\phi_1(0) - \phi_2(0)\|^2_{H^{-1}(\Omega)} + \|\psi_1(0) - \psi_2(0)\|^2_{L^2(\Gamma)}),\end{aligned} \tag{5.66}$$

where the positive constants c and K are independent of t.

Proof:
We use (5.63), with $\phi = \phi_1$ and $\varphi = \phi_2$, and we obtain:

$$\begin{aligned}\int_s^t ((A^{-1}\partial_t\phi_1, \phi_1-\phi_2)_\Omega + (\partial_t\phi_1, \phi_1-\phi_2)_\Gamma)\mathrm{d}\tau \\ + \int_s^t (B(\phi_2, \phi_1-\phi_2)_\Omega + (f(\phi_2), \phi_1-\phi_2)_\Omega)\mathrm{d}\tau \\ \leqslant \int_s^t \left(L(\phi_1, A^{-1}(\phi_1-\phi_2))_\Omega - (g(\phi_1), \phi_1-\phi_2)_\Gamma\right)\mathrm{d}\tau,\end{aligned} \tag{5.67}$$

and (5.65) with $\phi = \phi_2$ and $\varphi = \phi_1$, we find:

$$\begin{aligned}\int_s^t ((A^{-1}\partial_t\phi_2, \phi_2-\phi_1)_\Omega + (\partial_t\phi_2, \phi_2-\phi_1)_\Gamma)\mathrm{d}\tau \\ + \int_s^t (B(\phi_2, \phi_2-\phi_1)_\Omega + (f(\phi_2), \phi_2-\phi_1)_\Omega)\mathrm{d}\tau \\ \leqslant \int_s^t \left(L(\phi_2, A^{-1}(\phi_2-\phi_1))_\Omega - (g(\phi_2), \phi_2-\phi_1)_\Gamma\right)\mathrm{d}\tau.\end{aligned} \tag{5.68}$$

Summing the two resulting inequalities (5.67) and (5.68) and using the fact that (ϕ_i, ψ_i) are absolutely continuous on $[s,t],\ i=1,2$, with values in $H^{-1}(\Omega)\times L^2(\Gamma)$, we obtain:

$$\begin{aligned}\frac{1}{2}(&\|(\phi_1(t),\psi_1(t))-(\phi_2(t),\psi_2(t))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)}\\ &\qquad-\|(\phi_1(s),\psi_1(s))-(\phi_2(s),\psi_2(s))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)})\\ &\leqslant\int_s^t(L\|\phi_1(\tau)-\phi_2(\tau)\|^2_{H^{-1}(\Omega)}-(g(\phi_1(\tau))-g(\phi_2(\tau)),\phi_1(\tau)-\phi_2(\tau)))_\Gamma \mathrm{d}\tau.\end{aligned} \tag{5.69}$$

Using the fact that g is bounded globally and applying the Gronwall inequality to (5.69), we have:

$$\begin{aligned}&\|(\phi_1(t),\psi_1(t))-(\phi_2(t),\psi_2(t))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)}\\ &\qquad\leqslant ce^{K(t-s)}\|(\phi_1(s),\psi_1(s))-(\phi_2(s),\psi_2(s))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)},\end{aligned}$$

where the positive constants c and K are independent of $t>s>0$ and $(\phi_i,\psi_i),\ i=1,2$. Passing to the limit as $s\to 0$ and thanks to the continuity of $(\phi_i,\psi_i),\ i=1,2$ from Definition 5.3.1, condition 4, we get the desired estimate, which in particular gives the uniqueness.
Now, we need to prove that the above definition of a solution is independent of the choice of L. To do so, we assume that (ϕ_1,ψ_1) is a variational solution for $L=L_1$ and (ϕ_2,ψ_2) is a variational solution for $L=L_2$. Using the following relation:

$$\begin{aligned}&B_{L_1}(\phi_2,\phi_1-\phi_2)-B_{L_2}(\phi_2,\phi_1-\phi_2)\\ &\quad=L_1(\phi_1,A^{-1}(\phi_1-\phi_2))_\Omega-L_2(\phi_1,A^{-1}(\phi_1-\phi_2))_\Omega-(L_1-L_2)\|\phi_1-\phi_2\|^2_{H^{-1}(\Omega)},\end{aligned}$$

and arguing as in the proof of (5.66), we find:

$$\begin{aligned}\int_s^t((A^{-1}(\partial_t\phi_1-\partial_t\phi_2),&\phi_1-\phi_2)_\Omega+(\partial_t\phi_1-\partial_t\phi_2,\phi_1-\phi_2)_\Gamma)\mathrm{d}\tau\\ &+\int_s^t(B_{L_1}(\phi_2,\phi_1-\phi_2)_\Omega-B_{L_2}(\phi_2,\phi_1-\phi_2)_\Omega)\,\mathrm{d}\tau\\ \leqslant&L_1\int_s^t\big(\phi_1,A^{-1}(\phi_1-\phi_2)\big)_\Omega\,\mathrm{d}\tau+L_2\int_s^t\big(\phi_2,A^{-1}(\phi_2-\phi_1)\big)_\Omega\,\mathrm{d}\tau\\ &-\int_s^t(g(\phi_1(\tau))-g(\phi_2(\tau)),\phi_1(\tau)-\phi_2(\tau))_\Gamma\,\mathrm{d}\tau.\end{aligned} \tag{5.70}$$

After simplification, (5.70) gives:

$$\begin{aligned}\frac{1}{2}(&\|(\phi_1(t),\psi_1(t))-(\phi_2(t),\psi_2(t))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)}\\ &\qquad-\|(\phi_1(s),\psi_1(s))-(\phi_2(s),\psi_2(s))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)})\end{aligned}$$

$$\leqslant\int_s^t(L_1\|\phi_1(\tau)-\phi_2(\tau)\|^2_{H^{-1}(\Omega)}-(g(\phi_1(\tau))-g(\phi_2(\tau)),\phi_1(\tau)-\phi_2(\tau)))_\Gamma\mathrm{d}\tau. \tag{5.71}$$

which coincides with (5.69) and also leads to (5.66). Theorem 5.3.2 is thus proven. □

Theorem 5.3.3. *For every initial data $(\phi_0, \psi_0) \in \Phi$, problem (5.58) possesses a unique variational solution (ϕ, ψ) in the sense of Definition 5.3.1. Such a solution regularizes as $t > 0$ and all the uniform estimates obtained above hold. In particular, the following estimate is valid for every $\delta > 0$ and $t > 0$:*

$$\begin{aligned}\|\phi(t)\|^2_{C^\alpha(\Omega)} + \|\phi(t)\|^2_{H^2(\Omega_\delta)} + \|\psi(t)\|^2_{H^2(\Gamma)} + \|\phi(t)\|^2_{H^1(\Omega)} + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)}\\ +\|\partial_t\psi(t)\|^2_{L^2(\Gamma)} + \|\nabla D_\tau\phi(t)\|^2_{[L^2(\Omega)]^6} + \|f(\phi(t))\|_{L^1(\Omega)}\\ + \int_t^{t+1}(\|\partial_t\phi(s)\|^2_{H^1(\Omega)} + \|\partial_t\psi(s)\|^2_{H^1(\Gamma)})\mathrm{d}s\\ \leqslant C(1 + \|\phi(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)})e^{-\gamma t},\end{aligned} \tag{5.72}$$

for some positive constants α and C which are independent of t and ϕ, where D_τ denotes the tangential part of the gradient ∇.

Proof:
Repeating the derivation of the variational inequality (5.63), we obtain that (ϕ_N, ψ_N) satisfies:

$$\begin{aligned}\int_s^t (A^{-1}\partial_t\phi_N, \phi_N - \varphi)_\Omega + (\partial_t\phi_N, \phi_N - \varphi)_\Gamma + B(\varphi, \phi_N - \varphi)_\Omega + (f_N(\varphi), \phi_N - \varphi)_\Omega \mathrm{d}\tau\\ \leqslant \int_s^t \left(L(\phi_N, A^{-1}(\phi_N - \varphi))_\Omega - (g(\phi_N), \phi_N - \varphi)_\Gamma\right)\mathrm{d}\tau,\end{aligned} \tag{5.73}$$

for every admissible test function φ and every $t > s > 0$. Our aim is to pass to the limit $N \to +\infty$.
We start with the case when the initial datum ϕ_0 is smooth and satisfies the additional conditions:

$$|\phi_0(x)| \leqslant 1 - \delta,\ \delta > 0,\ \psi_0 := u_0|_\Gamma. \tag{5.74}$$

Then, by (5.47), we have:

$$\|\phi_N(t)\|_{L^\infty([0,T],H^1(\Omega))} + \|\phi_N(t)\|_{L^\infty([0,T],H^2(\Omega_\delta))} + \|\psi_N(t)\|_{L^\infty([0,T],H^2(\Gamma))} \leqslant C,$$

$$\begin{aligned}\|\partial_t\phi_N(t)\|_{L^\infty([0,T],H^{-1}(\Omega))} + \|\partial_t\psi_N(t)\|_{L^\infty([0,T],L^2(\Gamma))} + \|\partial_t\phi_N(t)\|_{L^2([0,T],H^1(\Omega))}\\ + \|\partial_t\psi_N(t)\|_{L^2([0,T],H^1(\Gamma))} \leqslant C,\end{aligned} \tag{5.75}$$

$$\|D^2_\tau\phi_N(t)\|_{L^\infty([0,T],L^2(\Omega))} \leqslant C,$$

$$\|\phi_N(t)\|_{L^\infty([0,T],C^\alpha(\Omega))} \leqslant C, \tag{5.76}$$

where the positive constant C depends on Ω, Γ, ϕ_0, ψ_0 and T but is independent of N and t. From this point on, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not relabeled. Thus, we observe that weak and weak star compactness results applied to the sequence ϕ_N entail that there exists a function ϕ such that as $N \to \infty$, the following properties hold:

$$\phi_N \to \phi \text{ weakly-* in } L^\infty\left([0,T],(H^1(\Omega)\otimes H^2(\Gamma))\cap H^2(\Omega_\delta)\right),$$

$$(\partial_t\phi_N,\partial_t\psi_N)\to(\partial_t\phi,\partial_t\psi) \text{ weakly-* in } L^\infty\left([0,T],(H^{-1}(\Omega)\times L^2(\Gamma))\right),$$

$$(\partial_t\phi_N,\partial_t\psi_N)\to(\partial_t\phi,\partial_t\psi) \text{ weakly in } L^2\left([0,T],(H^1(\Omega)\otimes H^1(\Gamma))\right),$$

$$D_\tau^2\phi_N \to D_\tau^2\phi \text{ weakly-* in } L^\infty([0,T],L^2(\Omega)).$$

It follows from (5.75) and (5.76), using the compactness theorem of Aubin-Lions, that:

$$\phi_N \to \phi \text{ strongly in } C^\gamma([0,T]\times\Omega) \text{ for some } \gamma>0.$$

These convergence results allow us to pass to the limit $N \to +\infty$ in (5.73) and prove that the limit function satisfies (5.63) for any admissible test function φ. The only nontrivial term containing the nonlinearity f_N can be treated by using the inequality $|f_N(\varphi)| \leqslant |f(\varphi)|$, the fact that $f(\varphi) \in L^1([0,T]\times\Omega)$ and the Lebesgue dominated convergence theorem. The crucial point $-1 < \phi(t,x) < 1$, for almost all $(t,x) \in \mathbb{R}\times\Omega$, can be proven as in Miranville & Zelik [45]. Indeed, taking into account the definition of f_N and the fact that the L^1-norm of $f_N(\phi_N)$ is uniformly bounded, we can conclude:

$$\text{meas}\,\{(t,x)\in[T,T+1]\times\Omega,\ |\phi_M(t,x)|>1+1/N\}\leqslant\pi(1/N),\ M\geqslant N, \quad (5.77)$$

where

$$\pi(x) := \frac{C}{\max\{|f(1-x)|,|f(x-1)|\}}, \quad (5.78)$$

for some positive constant C which is independent of $T\in\mathbb{R}^+$, of N and M, with $M \geqslant N$. Using the fact that $\pi(x)\to 0$ as $x\to 0$ and passing to the limit $M,\ N\to+\infty$ in (5.77), we conclude that:

$$\text{meas}\,\{(t,x)\in[T,T+1]\times\Omega,\ |\phi(t,x)|=1\}=0, \quad (5.79)$$

so that

$$|\phi(t,x)|<1 \text{ for almost all } (t,x)\in\mathbb{R}^+\times\Omega. \quad (5.80)$$

Inequality (5.80) and the convergence $\phi_N\to\phi$ strongly in $C^\gamma([0,T]\times\Omega)$ imply the almost everywhere convergence $f_N(\phi_N)\to f(\phi)$. Therefore, Fatou's lemma gives:

$$\|f(\phi)\|_{L^1([0,T]\times\Omega)} \leqslant \liminf_{N\to+\infty}\|f_N(\phi_N)\|_{L^1([0,T]\times\Omega)} < +\infty. \quad (5.81)$$

Thus, $f(\phi)\in L^1([0,T]\times\Omega)$ and (ϕ,ψ) is a variational solution to problem (5.58). In particular, the L^1-estimate on $f(\phi)$ follows from (5.81). Since the separation from singularities is not ensured on the boundary, we are not allowed to pass to the limit in $\|F_N(\phi_N(t))\|_{L^1(\Gamma)}$.

Finally, we remove assumption (5.74). In that case, we approximate the initial

datum $(\phi_0, \psi_0) \in \Phi$ by a sequence (ϕ_0^k, ψ_0^k) of smooth functions satisfying (5.74) such that:

$$\|\phi_0 - \phi_0^k\|_{L^2(\Omega)} \to 0, \ \|\psi_0 - \psi_0^k\|_{L^2(\Gamma)} \to 0, \ \text{ as } k \to +\infty. \tag{5.82}$$

Let $(\phi_k(t), \psi_k(t))$ be a sequence of variational solutions of problem (5.58) satisfying $(\phi_k(0), \psi_k(0)) = (\phi_0^k, \psi_0^k)$, where $\phi_k|_\Gamma = \psi_k$. The existence of such a sequence of solutions was proved above. Then, by estimate (5.66) and assumption (5.82), we can see that (ϕ_k, ψ_k) is a Cauchy sequence in $C([0,T], H^{-1}(\Omega)\times L^2(\Gamma))$ and therefore, the limit function exists and

$$(\phi, \psi) := \lim_{k\to+\infty} (\phi_k, \psi_k) \in C([0,T], H^{-1}(\Omega) \times L^2(\Gamma)).$$

Then, the proof of the theorem is finished as above. □
We also have the following result:

Lemma 5.3.4. *Let $(\phi(t), \psi(t))$ be a variational solution of problem (5.58). Then, $\psi(t) = \phi(t)|_\Gamma$ for $t > 0$. Moreover, this solution solves (5.58) in the usual sense, that is, for any $\varphi \in C_0^\infty((0,T)\times\Omega)$, the following equation holds:*

$$\begin{aligned}\int_{\mathbb{R}^+} (A^{-1}\partial_t\phi(t), \varphi(t))_\Omega dt = &\int_{\mathbb{R}^+} ((\Delta\phi(t), \varphi(t))_\Omega - (f(\phi(t), \varphi(t))_\Omega)\, dt \\ &+ \int_{\mathbb{R}^+} \lambda(\phi(t), \varphi(t))_\Omega dt.\end{aligned} \tag{5.83}$$

Furthermore,

$$\phi \in L^\infty([\tau, T], W^{2,1}(\Omega)), \ 0 < \tau < T, \tag{5.84}$$

so that the trace of the normal derivative on the boundary,

$$[\partial_n\phi]_{int} := \partial_n\phi|_\Gamma \in L^\infty([\tau,T], L^1(\Gamma)), \ 0 < \tau < T, \tag{5.85}$$

exists.

Proof:
Since ϕ_N is uniformly bounded in $L^\infty([\tau,T], H^2(\Omega_\delta))$, $\forall\delta > 0$, and f_N is uniformly continuous, the sequence $f_N(\phi_N)$ is also uniformly bounded in $L^\infty([\tau,T], H^2(\Omega_\delta))$. Using this fact and that $f_N(\phi_N) \to f(\phi)$ a.e., we obtain using a weak version of the dominated convergence theorem that $f_N(\phi_N) \to f(\phi)$ weakly in $L^2([\tau,T], H^2(\Omega_\delta))$. Thus, we are allowed to pass to the limit in the equation corresponding to (5.83) for ϕ_N. We deduce from (5.83), that ϕ is a solution for:

$$A^{-1}\partial_t\phi(t) - \Delta\phi(t) + f(\phi(t)) - \lambda\phi(t) = 0, \ \text{ in } L^2_{loc}((\tau,T)\times\Omega_\delta).$$

Moreover, since $f(\phi)$ and $A^{-1}\partial_t\phi$ belong to $L^\infty((\tau,T), L^1(\Omega))$, we find that

$$\Delta\phi \in L^\infty((\tau,T), L^1(\Omega)).$$

Having the control of $\nabla D_\tau\phi$, we deduce that $\phi \in L^\infty((\tau,T), W^{2,1}(\Omega))$, which yields the existence of the trace (5.85). □

Concerning the second equation from (5.58), we use Theorem 5.2.6 and we see that:

$$\|\partial_t \psi_N\|_{L^\infty([\tau,T],L^2(\Gamma))} + \|\psi_N\|_{L^\infty([\tau,T],H^2(\Gamma))} \leqslant c, \tag{5.86}$$

where the constant c is independent of N. We have:

$$\partial_n \phi_N = \partial_t \psi_N - \Delta_\Gamma \psi_N + g(\psi_N). \tag{5.87}$$

Using (5.86), we deduce that $\partial_n \phi_N \in L^\infty([\tau,T],L^2(\Gamma))$. Passing to the limit as $N \to +\infty$, we have the weak-star convergence in $L^\infty([\tau,T],L^2(\Gamma))$

$$[\partial_n \phi]_{ext} := \lim_{N\to+\infty} \partial_n \phi_N|_\Gamma \in L^\infty([\tau,T],L^2(\Gamma)),\ T > \tau > 0, \tag{5.88}$$

and

$$\partial_t \psi - \Delta_\Gamma \psi + g(\psi) + [\partial_n \phi]_{ext} = 0,\ \text{on } \Gamma,\ T > \tau > 0.$$

In order to verify that the variational solution (ϕ, ψ) satisfies equations (5.58) in the usual sense, there only remains to check that:

$$[\partial_n \phi]_{int} = [\partial_n \phi]_{ext},\ \text{for almost every } (t,x) \in \mathbb{R}^+ \times \Gamma. \tag{5.89}$$

5.4 Additional regularity results and separation from the singularities

In this section, we formulate several sufficient conditions which ensure that every variational solution satisfies equation (5.58) in the usual sense. We have the following result which gives an additional regularity on ϕ close to the points where $|\phi(t,x)| < 1$.

Proposition 5.4.1. *Let the assumptions of Theorem 5.3.2 hold and let (ϕ, ψ) be a variational solution to* (5.58)*. For any δ, $T > 0$, we set:*

$$\Omega_\delta(T) = \{x \in \Omega, |\phi(T,x)| < 1 - \delta\}.$$

Then, $\phi \in H^2(\Omega_\delta(T))$ and the following estimate holds:

$$\|\phi\|_{H^2(\Omega_\delta(T))} \leqslant Q_{\delta,T}, \tag{5.90}$$

where the positive constant $Q_{\delta,T}$ depends on T and δ but is independent of the concrete choice of the solution ϕ.

Proof:
Since the solution $\phi(T,x)$ is Hölder continuous with respect to x, there exists a smooth nonnegative cut-off function $\theta(x)$ such that:

$$\begin{cases} \theta(x) \equiv 1,\ x \in \Omega_\delta(T), \\ \theta(x) \equiv 0,\ x \in \Omega \backslash \Omega_{\delta/2}(T), \\ \|\theta\|_{C^2(\mathbb{R}^3(\Omega))} \leqslant K_{\delta,T}, \end{cases} \tag{5.91}$$

where $K_{\delta,T}$ is independent of the concrete choice of the solution ϕ. Let $\phi_N(t,x)$ be a sequence of approximate solutions of problem (5.2) which converges to the

variational solution $\phi(t,x)$ as $N \to +\infty$. Then, since the convergence holds in the space $C^{\gamma}([0,T]\times\Omega)$ for some $\gamma > 0$, we have:

$$|\phi_N(T,x)| < 1 - \delta/4,\ x \in \Omega_{\delta/2}(T), \tag{5.92}$$

for N large enough. Setting $v_N := \theta(x)\phi_N(T,x)$ and $w_N := \theta(x)\psi_N(T,x)$, we have:

$$\begin{cases} \Delta_x\phi_N - f_N(\phi_N(T)) - \phi_N = h_1(T) := -\phi_N + \lambda\phi_N + A^{-1}\partial_t\phi_N, \\ \Delta_\Gamma\psi_N - \psi_N - \partial_n\phi_N = h_2(T) := g_0(\psi_N(T)) + \partial_t\psi_N. \end{cases} \tag{5.93}$$

Multiplying the first equation of (5.93) by θ, we find:

$$\begin{aligned} &\theta\Delta_x\phi_N(T) - \theta\phi_N(T) = \theta h_1(T) + \theta f_N(\phi_N(T)) \\ &\iff \Delta_x(\theta\phi_N(T)) - \theta\phi_N(T) = \theta h_1(T) + \theta f_N(\phi_N(T)) + 2\nabla_x\theta\cdot\nabla_x\phi_N(T) \\ &+ \phi_N(T)\Delta_x\theta \\ &\iff \Delta_x v_N - v_N = \theta h_1(T) + \theta f_N(\phi_N(T)) + 2\nabla_x\theta\cdot\nabla_x\phi_N(T) + \phi_N(T)\Delta_x\theta. \end{aligned} \tag{5.94}$$

Now, multiplying the second equation of (5.93) by θ, we obtain:

$$\begin{aligned} &\theta\Delta_\Gamma\psi_N(T) - \theta\psi_N(T) - \theta\partial_n\phi_N(T) = \theta h_2(T) \\ &\iff \Delta_\Gamma w_N - w_N - \partial_n w_N = \theta h_2(T) + 2\nabla_\Gamma\theta\cdot\nabla_\Gamma\psi_N(T) + \psi_N(T)\Delta_\Gamma\theta \\ &- \psi_N(T)\partial_n\theta. \end{aligned} \tag{5.95}$$

Thus, φ_N satisfies the following elliptic boundary value problem:

$$\begin{cases} \Delta_x v_N - v_N = \tilde{h_1}(\phi_N) := \theta h_1(T) + \theta f_N(\phi_N(T)) + 2\nabla_x\theta\cdot\nabla_x\phi_N(T) + \phi_N(T)\Delta_x\theta, \\ \Delta_\Gamma w_N - w_N - \partial_n w_N = \tilde{h_2}(\psi_N) := \theta h_2(T) + 2\nabla_\Gamma\theta\cdot\nabla_\Gamma\psi_N(T) + \psi_N(T)\Delta_\Gamma\theta \\ \qquad\qquad - \psi_N(T)\partial_n\theta. \end{cases} \tag{5.96}$$

Using the estimates (5.33), (5.49), (5.91) and (5.92), we find:

$$\|\tilde{h_1}(\phi_N)\|_{L^2(\Omega)} + \|\tilde{h_2}(\psi_N)\|_{L^2(\Gamma)} \leqslant Q_{\delta,T},$$

where the positive constant $Q_{\delta,T}$ is independent of N and of the concrete choice of the solution ϕ. Applying an H^2-regularity result to problem (5.96) (see [42]), we deduce that:

$$\|\phi_N(T)\|_{H^2(\Omega_\delta(T))} \leqslant Q_{\delta,T}. \tag{5.97}$$

By passing to the limit as $N \to +\infty$, we deduce that $\phi(T) \in H^2(\Omega_\delta(T))$ and that $\|\phi(T)\|_{H^2(\Omega_\delta(T))} \leqslant Q_{\delta,T}$. □

Lemma 5.4.2. *Let (ϕ,ψ) a variational solution to (5.58). Assume, in addition, that we have:*

$$|\phi(t_0,x_0)| < 1, \tag{5.98}$$

for some $(t_0,x_0) \in (0,+\infty)\times\Gamma$. Then, there exists a neighborhood $(t_0-\varepsilon,t_0+\varepsilon)\times V$ of (t_0,x_0) in $\mathbb{R}\times\Gamma$ such that:

$$[\partial_n\phi]_{int}(t,x) = [\partial_n\phi]_{ext}(t,x),\ \forall (t,x) \in (t_0-\varepsilon,t_0+\varepsilon)\times V. \tag{5.99}$$

In particular, if ϕ satisfies:

$$|\phi(t,x)| < 1 \text{ for almost all } (t,x) \in \mathbb{R}^+ \times \Gamma, \tag{5.100}$$

then, the equality $[\partial_n \phi]_{int} = [\partial_n \phi]_{ext}$ holds almost everywhere in $(0,+\infty) \times \Gamma$ and (ϕ, ψ) solves (5.58) in the usual sense.

Proof:
We know that ϕ is Hölder continuous with respect to t and x. Thus there exists $\varepsilon > 0$ such that $|\phi(t,x)| \leqslant 1 - \varepsilon$ holds for all (t,x) in a neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon) \times V_\varepsilon$ of (t_0, x_0) in $(0,+\infty) \times \Omega$. Thanks to Proposition 5.4.1, the approximate solution ϕ_N (converging to ϕ) satisfies:

$$\|\phi_N\|_{L^\infty([t_0-\varepsilon,t_0+\varepsilon],H^2(V_\varepsilon))} \leqslant C,$$

where the positive constant C is independent of N. Then, we can assume that $\phi_N \to \phi$ weakly-* in this space, which yields that $\partial_n \phi_N|_\Gamma \to \partial_n \phi|_\Gamma$ weakly in $L^2([t_0 - \varepsilon, t_0 + \varepsilon] \times V)$ for some proper neighborhood V of x_0. This convergence result, together with the definition (5.88), leads to equality (5.99) and relation (5.100) is a consequence of (5.99). □
Thus, in order to prove that any variational solution ϕ is a solution in the usual sense, it is sufficient to verify that ϕ satisfies (5.100).

Corollary 5.4.3. *Let the assumptions of Theorem 5.3.2 hold. We assume that:*

$$\lim_{s \to \pm 1} F(s) = +\infty. \tag{5.101}$$

Then, for every variational solution ϕ of problem (5.58), relation (5.100) holds and the potential F verifies:

$$F(\phi(t)) \in L^1(\Gamma) \text{ and } \|F(\phi(t))\|_{L^1(\Gamma)} \leqslant C_T, \tag{5.102}$$

for almost all $t \geqslant T > 0$.

Proof:
Let ϕ_N be a sequence of approximate solutions converging to the variational solution ϕ. Applying estimate (6.4) in Miranville & Zelik [45], we obtain:

$$\|F_N(\phi_N)\|_{L^1(\Gamma)} \leqslant C_T, \ t \geqslant T, \tag{5.103}$$

where the constant C_T is independent of N. Arguing as in the proof of Theorem 5.3.3, we obtain that:

$$\text{meas } \{(t,x) \in [T, T+1] \times \Gamma, \ |\phi_M(t,x)| \geqslant 1 - 1/N\} \leqslant \pi(1/N),$$

where

$$\pi(x) = \frac{C}{\max\{|F(1-x)|, |F(x-1)|\}}.$$

Passing to the limit $M, N \to +\infty$, we deduce that:

$$\text{meas } \{(t,x) \in [T, T+1] \times \Gamma, \ |\phi(t,x)| = 1\} = 0.$$

Thus, condition (5.100) holds.
Then, using the convergence $\phi_N \to \phi$ in $C^\gamma([0,T]\times\Omega)$, with $\gamma > 0$ and the already proved statement 5.100, we conclude that $F_N(\phi_N) \to F(\phi)$ almost everywhere in $\mathbb{R}^+ \times \Gamma$ and therefore, thanks to the Fatou lemma,

$$\|F(\phi)\|_{L^1(\Gamma)} \leqslant \liminf_{N\to+\infty} \|F_N(\phi_N)\|_{L^1(\Gamma)} \leqslant C_T.$$

□

Corollary 5.4.4. *Let the assumptions of Theorem 5.3.2 hold. We assume that:*

$$g(-1) + \varepsilon \leqslant 0 \leqslant g(1) - \varepsilon, \tag{5.104}$$

for some $\varepsilon > 0$. Then, for every variational solution ϕ of problem (5.58)*, estimate* (5.100) *holds and*

$$\|f(\phi)\|_{L^1([t,t+1]\times\Gamma)} \leqslant C_{\varepsilon,T},\ t \geqslant T > 0, \tag{5.105}$$

where the constant $C_{\varepsilon,T}$ is independent of the concrete choice of the variational solution ϕ.

Proof:
We consider the nonlinear elliptic-parabolic system:

$$\begin{cases} \Delta\phi_N(t) - f_N(\phi_N(t)) - \phi_N(t) = h_1(t), \\ \phi_N|_\Gamma = \psi_N, \\ \partial_t\psi_N(t) - \Delta_\Gamma\psi_N(t) + \partial_n\phi_N(t) + g(\psi_N(t)) = 0. \end{cases} \tag{5.106}$$

Arguing as in Miranville & Zelik [45], we have:

$$g(s)\cdot f_N(s) \geqslant \frac{\varepsilon}{2}|f_N(s)| + C_\varepsilon,\ s \in \mathbb{R}, \tag{5.107}$$

where the constant C_ε depends on g and ε but is independent of N. To derive (5.105), we multiply (5.106) by $f_N(\phi_N)$ and we use (5.107). We obtain:

$$\begin{aligned} &\frac{d}{dt}\int_\Gamma F_N(\phi_N(t))\mathrm{d}s + (f_N'(\phi_N(t))\nabla\phi_N(t), \nabla\phi_N(t))_\Omega \\ &+ (f_N'(\phi_N(t))\nabla_\Gamma\phi_N(t), \nabla_\Gamma\phi_N(t))_\Gamma + 1/2\|f_N(\phi_N(t))\|^2_{L^2(\Omega)} + \varepsilon/2\|f_N(\phi_N(t))\|_{L^1(\Gamma)} \\ &\leqslant C_\varepsilon(1 + \|h_1(t)\|^2_{L^2(\Omega)}). \end{aligned} \tag{5.108}$$

The L^2-norm of $h_1(t)$ is controlled thanks to (5.32) and we find:

$$\begin{aligned} \|h_1(t)\|^2_{L^2(\Omega)} &\leqslant c(1 + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)}) \\ &\leqslant c(1 + \|\partial_t\phi(0)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(0)\|^2_{L^2(\Gamma)}) \\ &< +\infty. \end{aligned}$$

Integrating (5.108) with respect to t and using the fact that $f'_N \geqslant 0$, we find:

$$\begin{aligned}\int_\Gamma F_N(\phi_N(t+1))d\sigma + \int_t^{t+1} (\|\nabla\phi_N(s)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\phi_N(s)\|^2_{L^2(\Gamma)}\mathrm{d}s \\ + \int_t^{t+1} (\|f_N(\phi_N(s))\|^2_{L^2(\Omega)}) + \varepsilon/2\|f_N(\phi_N(s))\|_{L^1(\Gamma)})\mathrm{d}s \\ \leqslant C_{\varepsilon,T} + \int_\Gamma F_N(\phi_N(t))d\sigma.\end{aligned} \tag{5.109}$$

We deduce from (5.109):

$$\begin{aligned}\|f_N(\phi_N)\|_{L^1([t,t+1]\times\Gamma)} &\leqslant 2/\varepsilon(\|F_N(\phi_N(t))\|_{L^1(\Gamma)} + \|F_N(\phi_N(t+1))\|_{L^1(\Gamma)}) + C_{\varepsilon,T} \\ &\leqslant C'_{\varepsilon,T}.\end{aligned}$$

Arguing as in Corollary 5.4.3, we finish the proof. □

5.5 Attractors and exponential attractors

In this section, we study the asymptotic behavior of the system. We denote by $\Phi^w := H^{-1}(\Omega) \times L^2(\Gamma)$. The space Φ^w is endowed with the natural norm:

$$\|\varphi\|^2_{\Phi^w} = \|\varphi\|^2_{H^{-1}(\Omega)} + \|\varphi\|^2_{L^2(\Gamma)}, \text{ for all } \varphi \in \Phi^w.$$

We endow the phase space Φ with the metric of Φ^w.
We have the following result:

Corollary 5.5.1. *Under the assumptions of Theorem 5.3.3, equation* (5.58) *generates a solution semigroup $S(t) : \Phi \to \Phi$, where $S(t)(\phi_0, \psi_0) := (\phi(t), \psi(t))$ is the unique variational solution of problem* (5.58) *departing from (ϕ_0, ψ_0). Furthermore, we have the following Lipschitz continuity property in the Φ^w- topology:*

$$\begin{aligned}&\|S(t)(\phi_0^1, \psi_0^1) - S(t)(\phi_0^2, \psi_0^2)\|^2_{\Phi^w} \\ &\qquad + \int_t^{t+1} \|S(s)(\phi_0^1, \psi_0^1) - S(s)(\phi_0^2, \psi_0^2)\|^2_{H^1(\Omega)\times H^1(\Gamma)}\mathrm{d}s \\ &\leqslant Ce^{Kt}\|(\phi_0^1 - \phi_0^2, \psi_0^1 - \psi_0^2))\|^2_{\Phi^w},\end{aligned} \tag{5.110}$$

for all $(\phi_0^1, \psi_0^1), (\phi_0^2, \psi_0^2) \in \Phi$.

This corollary is a direct consequence of Proposition 5.2.7.
The following proposition gives the existence of the global attractor $\mathcal{A}$ for this semigroup. We recall that, by definition, a set $\mathcal{A} \subset \Phi$ is the global attractor for the semigroup $S(t)$ if the following properties are satisfied:

1. It is a compact subset of Φ;
2. It is strictly invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$, $\forall t > 0$;

3. It attracts Φ as $t \to +\infty$, i.e., for every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in Φ, there exists a time $T = T(\mathcal{O})$ such that:

$$S(t)X \subset \mathcal{O}(\mathcal{A}), \;\; t \geqslant T.$$

Proposition 5.5.2. *The semigroup $S(t)$ associated with the variational solutions of problem (5.58) possesses the global attractor $\mathcal{A}$ which is bounded in the space $C^\alpha(\Omega) \times C^\alpha(\Gamma)$ for some positive constant $\alpha < 1/4$.*

Proof:
The semigroup $S(t)$ is dissipative. Indeed, thanks to estimate (5.27) there exists $R_0 > 0$ such that the ball $B_{\mathcal{H}_1}(R_0)$ centered on zero with radius R_0 in $H^1(\Omega)\times H^1(\Gamma)$ is absorbing in Φ and compact in the topology of Φ^w. In particular, there exists a time $t_0 \geqslant 1$ such that $S(t)B_{\mathcal{H}_1}(R_0) \subset B_{\mathcal{H}_1}(R_0)$, for any $t \geqslant t_0$. As a consequence, the set:

$$\mathbb{B}_0 := \overline{\cup_{t\geqslant t_0} S(t)B_{\mathcal{H}_1}(R_0)}^{\Phi^w} \tag{5.111}$$

is absorbing and positively invariant. Thus the existence of the global attractor $\mathcal{A}$ follows from a proper abstract attractor's existence theorem (see Temam [54]). □
In the following theorem, we prove the existence of an exponential attractor which by definition contains the global attractor and has finite fractal dimension. To do this, we first recall the definition of the exponential attractor where $\mathcal{A}$ is the global attractor for the semigroup $\{S(t)\}_{t\geqslant 0}$:

Definition 5.5.3. A set $\mathcal{M} \subset \Phi$ is called an exponential attractor for the semigroup $\{S(t)\}_{t\geqslant 0}$ if :

1. $\mathcal{M}$ is compact in Φ;
2. $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geqslant 0$;
3. $\mathcal{M}$ has finite fractal dimension, $d_F(\mathcal{M}) < \infty$;
4. $\mathcal{M}$ attracts Φ exponentially fast as $t \to +\infty$, i.e.,

$$dist_E(S(t)\Phi, \mathcal{M}) \leqslant Ce^{-ct}, \; \forall t \geqslant 0,$$

where the pseudo-distance *dist* is the standard Hausdorff pseudo-distance between two sets, defined by $dist_E(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_E$.

Theorem 5.5.4. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M}$ which is bounded in $C^\alpha(\Omega) \times C^\alpha(\Gamma)$, $\alpha < 1/4$.*

Proof:
There exists a positive constant $R = R(R_0)$ such that:

$$\begin{aligned}\|\phi(t)\|_{C^\alpha([t,t+1]\times\Omega)} + \|\phi(t)\|_{H^2(\Gamma)} + \|\partial_t\phi(t)\|_{H^{-1}(\Omega)} + \|\partial_t\phi(t)\|_{L^2(\Gamma)} \\ + \|f(\phi(t))\|_{L^1(\Omega)} + \|\partial_t\phi(t)\|_{L^2([t,t+1],H^1(\Omega))} + \|\partial_t\phi(t)\|_{L^2([t,t+1],H^1(\Gamma))} \leqslant R,\end{aligned} \tag{5.112}$$

for any initial datum in $\mathbb{B}_0$ where $\mathbb{B}_0$ is the set defined by (5.111). In particular, for every point $(\phi, \psi) \in \mathbb{B}_0$, there holds $\phi|_\Gamma = \psi$. We consider an arbitrary small $\varepsilon-$ball $B(\varepsilon, \phi_0; \Phi^w)$ in the space $\mathbb{B}_0$ endowed with the metric of Φ^w and centered on ϕ_0, where $0 < \varepsilon \leqslant \varepsilon_0 \ll 1$, with the parameter ε_0 that will be fixed below. Let also $\phi^0(t),\ t \geqslant 0$, be the solution starting from ϕ_0. We introduce the sets:

$$\Omega_\delta(\phi_0) := \{x \in \Omega,\ |\phi_0(x)| \leqslant 1 - \delta\},$$

$$\overline{\Omega}_\delta(\phi_0) := \{x \in \Omega,\ |\phi_0(x)| > 1 - \delta\},$$

where δ is a sufficiently small positive number. Then, thanks to the Hölder continuity of ϕ_0 with respect to x, we have for all $x_1 \in \partial\Omega_{\delta_1}(\phi_0),\ x_2 \in \partial\Omega_{\delta_2}(\phi_0)$:

$$0 < |\delta_1 - \delta_2| \leqslant |\phi_0(x_1) - \phi_0(x_2)| \leqslant c|x_1 - x_2|^\alpha.$$

Thus, there is a strict separation between $\partial\Omega_{\delta_1}(\phi_0)$ and $\partial\Omega_{\delta_2}(\phi_0)$ for any $\delta_1 \neq \delta_2$, i.e.

$$d(\partial\Omega_{\delta_1}(\phi_0), \partial\Omega_{\delta_2}(\phi_0)) \geqslant C_{\delta_1,\delta_2} > 0,\ \delta_1 \neq \delta_2, \tag{5.113}$$

where the constant C_{δ_1,δ_2} depends on $\delta_1,\ \delta_2$.
We note that, since $\phi^0(t)$ is uniformly Hölder continuous with respect to t and x, there exists $T = T(\delta)$ such that:

$$\begin{aligned} &|\phi^0(t)| \leqslant 1 - \delta/2,\ x \in \Omega_\delta(\phi_0),\ t \in [0, T], \\ &|\phi^0(t)| \geqslant 1 - 3\delta,\ x \in \overline{\Omega}_{2\delta}(\phi_0),\ t \in [0, T]. \end{aligned}$$

Furthermore, using again the uniform Hölder continuity, we have:

$$\begin{aligned} \|\phi_1(t) - \phi_2(t)\|_{C(\Omega)} \leqslant& C\|\phi_1(t) - \phi_2(t)\|^\kappa_{\Phi^w}\|\phi_1(t) - \phi_2(t)\|^{1-\kappa}_{C^\alpha(\Omega)} \\ \leqslant& C_T\|\phi_1(0) - \phi_2(0)\|^\kappa_{\Phi^w}\|\phi_1(t) - \phi_2(t)\|^{1-\kappa}_{C^\alpha(\Omega)} \\ \leqslant& C_T\varepsilon^\kappa, \end{aligned}$$

for every $\phi_1(0),\ \phi_2(0)$ in $B(\varepsilon, \phi_0, \Phi^w)$. We can fix $\varepsilon_0 = \varepsilon_0(\delta)$ such that:

$$\begin{aligned} &|\phi(t)| \leqslant 1 - \delta/4,\ x \in \Omega_\delta(\phi_0),\ t \in [0, T], \\ &|\phi(t)| \geqslant 1 - 4\delta,\ x \in \overline{\Omega}_{2\delta}(\phi_0),\ t \in [0, T], \end{aligned} \tag{5.114}$$

for all trajectories $\phi(t)$ starting from the ball $B(\varepsilon, \phi_0, \Phi^w),\ \varepsilon \leqslant \varepsilon_0$.
Due to (5.113), there exists a smooth cut-off function $\theta \in C^\infty(\mathbb{R}^3, [0, 1])$ such that:

$$\theta(x) = \begin{cases} 0, & \text{if } x \in \overline{\Omega}_\delta(\phi_0), \\ 1, & \text{if } x \in \Omega_{2\delta}(\phi_0). \end{cases}$$

Furthermore, θ satisfies the additional condition:

$$\|\theta\|_{C^k(\mathbb{R}^3)} \leqslant C_k,$$

where $k \in \mathbb{N}$ is arbitrary and the constant C_k depends on δ, but is independent of the choice of $\phi_0 \in \mathbb{B}_0$. The second estimate of (5.114) yields:

$$f'(\phi(t, x)) \geqslant \Lambda(\delta),\ x \in \overline{\Omega}_{2\delta}(\phi_0),\ t \in [0, T], \tag{5.115}$$

for all trajectories $\phi(t)$ starting from the ball $B(\varepsilon, \phi_0, \Phi^w)$, where

$$\Lambda(\delta) := \min\left\{f'(1-4\delta), f'(-1+4\delta)\right\}. \tag{5.116}$$

Indeed, for every $x \in \overline{\Omega}_{2\delta}(\phi_0)$, we have that $|\phi(t)| \geqslant 1-4\delta$. Then, using the fact that sgn $s \cdot f''(s) \geqslant 0$, we obtain:

$$\begin{array}{l} \times \text{if } \phi(t) \geqslant 1-4\delta \Rightarrow f''(\phi(t)) \geqslant 0 \Rightarrow f'(\phi(t)) \geqslant f'(1-4\delta), \\ \times \text{if } \phi(t) \leqslant -1+4\delta \Rightarrow f''(\phi(t)) \leqslant 0 \Rightarrow f'(\phi(t)) \geqslant f'(-1+4\delta), \end{array} \tag{5.117}$$

which yields (5.115). Since $f'(s) \xrightarrow[s\to\pm1]{} +\infty$, then, $\Lambda(\delta) \xrightarrow[\delta\to0]{} +\infty$ and we can fix $\delta > 0$ close enough to zero such that $\Lambda(\delta)$ is arbitrarily large. The next lemma gives some kind of smoothing property for the difference of two solutions.

Lemma 5.5.5. *Let the above assumptions hold. Then, there exists $\delta > 0$ such that:*

$$\|\phi_1(T) - \phi_2(T)\|^2_{\Phi^w} \leqslant e^{-\beta T}\|\phi_1(0) - \phi_2(0)\|^2_{\Phi^w} + C\int_0^T \|\theta\left(\phi_1(s) - \phi_2(s)\right)\|^2_{L^2(\Omega)}\mathrm{d}s, \tag{5.118}$$

where the positive constants β and C are independent of $\phi_1(0),\ \phi_2(0) \in B(\varepsilon, \phi_0, \Phi^w)$ and $\phi_0 \in \mathbb{B}_0$.

Proof:
We set $\phi(t) = \phi_1(t) - \phi_2(t)$. Then, ϕ solves the following problem:

$$\left\{\begin{array}{l} A^{-1}\partial_t\phi = \Delta\phi - l(t)\phi + \lambda\phi, \text{ in } \Omega, \\ \partial_n(\Delta\phi - l(t)\phi + \lambda\phi)|_\Gamma = 0, \\ \partial_t\phi - \Delta_\Gamma\phi + \partial_n\phi + \phi(t) + m(t)\phi = 0, \text{ on } \Gamma, \end{array}\right. \tag{5.119}$$

where

$$l(t) := \int_0^t f'(s\phi_1(t) + (1-s)\phi_2(t))\mathrm{d}s \text{ and } m(t) := \int_0^t g_0'(s\phi_1(t) + (1-s)\phi_2(t))\mathrm{d}s.$$

Multiplying (5.119) by $\phi(t)$ and integrating over Ω, we obtain:

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\phi(t)\|^2_{L^2(\Gamma)}) + \|\nabla\phi(t)\|^2_{L^2(\Omega)} \\ \qquad + \|\nabla_\Gamma\phi(t)\|^2_{L^2(\Gamma)} + \|\phi(t)\|^2_{L^2(\Gamma)} + (l(t)\phi(t), \phi(t))_\Omega \\ \qquad = \lambda\|\phi(t)\|^2_{L^2(\Omega)} - (m(t)\phi(t), \phi(t))_\Gamma \\ \qquad \leqslant \lambda\|\phi(t)\|^2_{L^2(\Omega)} + C_0\|\phi(t)\|^2_{L^2(\Gamma)}, \end{array} \tag{5.120}$$

where $C_0 = \|g'\|_{C([-1,1])}$.
Due to (5.115), we have:

$$\begin{array}{rl} \displaystyle\int_\Omega l(t,x)|\phi(t,x)|^2\mathrm{d}x \geqslant \int_{\overline{\Omega}_{2\delta}} l(t,x)|\phi(t,x)|^2\mathrm{d}x & \geqslant \Lambda\|\phi\|^2_{L^2(\overline{\Omega}_{2\delta})} \\ & = \Lambda\|\phi\|^2_{L^2(\Omega)} - \Lambda\|\phi\|^2_{L^2(\Omega_{2\delta})} \\ & \geqslant \Lambda\|\phi\|^2_{L^2(\Omega)} - \Lambda\|\theta\phi\|^2_{L^2(\Omega)}. \end{array} \tag{5.121}$$

Thus, we obtain:

$$\begin{aligned}\frac{d}{dt}\|\phi(t)\|^2_{\Phi^w} + 2\|\nabla\phi(t)\|^2_{L^2(\Omega)} + 2\|\phi(t)\|^2_{H^1(\Gamma)} + 2(\Lambda-\lambda)\|\phi\|^2_{L^2(\Omega)} \\ \leqslant 2C_0\|\phi(t)\|^2_{L^2(\Gamma)} + 2\Lambda\|\theta\phi\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.122}$$

For the first term in the right hand-side of (5.122), we use the following trace inequality:

$$2\|\phi\|^2_{L^2(\Gamma)} \leqslant 2C\|\phi\|_{H^1(\Omega)}\|\phi\|_{L^2(\Omega)} \leqslant \frac{C}{\sqrt{\Lambda-\lambda}}\|\phi\|^2_{H^1(\Omega)} + C\sqrt{\Lambda-\lambda}\|\phi\|^2_{L^2(\Omega)}, \tag{5.123}$$

and the fact that for some $\omega \in (0,2)$, we have:

$$2\|\nabla\phi(t)\|^2_{L^2(\Omega)} + 2\|\phi(t)\|^2_{H^1(\Gamma)} \geqslant \omega\|\phi(t)\|^2_{H^1(\Omega)} + \omega\|\phi(t)\|^2_{H^1(\Gamma)}. \tag{5.124}$$

Thus, fixing δ in such a way that $C_0C \leqslant \omega\dfrac{\sqrt{\Lambda-\lambda}}{2}$ and using (5.123) and (5.124) in (5.122), we find:

$$\begin{aligned}&\frac{d}{dt}\|\phi(t)\|^2_{\Phi^w} + \omega\|\phi(t)\|^2_{H^1(\Omega)} + \omega\|\phi(t)\|^2_{H^1(\Gamma)} + 2(\Lambda-\lambda)\|\phi(t)\|^2_{L^2(\Omega)} \\ &\qquad\leqslant \frac{C_0C}{\sqrt{\Lambda-\lambda}}\|\phi(t)\|^2_{H^1(\Omega)} + C_0C\sqrt{\Lambda-\lambda}\|\phi(t)\|^2_{L^2(\Omega)} + 2\Lambda\|\theta\phi(t)\|^2_{L^2(\Omega)} \\ &\qquad\leqslant \omega/2\|\phi(t)\|^2_{H^1(\Omega)} + \omega/2(\Lambda-\lambda)\|\phi(t)\|^2_{L^2(\Omega)} + 2\Lambda\|\theta\phi(t)\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.125}$$

Taking $\beta' > 0$ small enough, estimate (5.125) leads to:

$$\frac{d}{dt}\|\phi(t)\|^2_{\Phi^w} + \omega/2\|\phi(t)\|^2_{H^1(\Omega)} + \beta'\left(\|\phi(t)\|^2_{H^1(\Gamma)} + \|\phi(t)\|^2_{L^2(\Omega)}\right) \leqslant 2\Lambda\|\theta\phi(t)\|^2_{L^2(\Omega)}. \tag{5.126}$$

Using the inequalities $\|\phi\|_{H^{-1}(\Omega)} \leqslant C\|\phi\|_{L^2(\Omega)}$ and $\|\phi\|_{L^2(\Gamma)} \leqslant C\|\phi\|_{H^1(\Gamma)}$, we end up with:

$$\frac{d}{dt}\|\phi(t)\|^2_{\Phi^w} + \omega/2\|\phi(t)\|^2_{H^1(\Omega)} + \beta\left(\|\phi(t)\|^2_{L^2(\Gamma)} + \|\phi(t)\|^2_{H^{-1}(\Omega)}\right) \leqslant 2\Lambda\|\theta\phi(t)\|^2_{L^2(\Omega)}, \tag{5.127}$$

for some positive constant β. Applying the Gronwall lemma, we obtain:

$$\begin{aligned}\|\phi(T)\|^2_{\Phi^w} &\leqslant \|\phi(0)\|^2_{\Phi^w}e^{-\beta T} + C\int_0^T e^{-\beta(T-s)}\|\theta\phi(s)\|^2_{L^2(\Omega)}\mathrm{d}s \\ &\leqslant \|\phi(0)\|^2_{\Phi^w}e^{-\beta T} + C\int_0^T \|\theta\phi(s)\|^2_{L^2(\Omega)}\mathrm{d}s,\end{aligned}$$

and estimate (5.118) is proven. □

Lemma 5.5.6. *Let the nonlinearities f and g satisfy the assumptions of Section 2. Then, there exists positive constants C and K independent of $\phi_i(0)$ in $B(\varepsilon, \phi_0, \Phi^w)$, $i = 1, 2$, and ϕ_0 in $\mathbb{B}_0$ such that the following estimate holds:*

$$\begin{aligned}\|\partial_t(\theta(\phi_1-\phi_2))\|_{L^2([0,T],H^{-3}(\Omega))} + \|\theta(\phi_1-\phi_2)\|_{L^2([0,T],H^1(\Omega))} \\ \leqslant Ce^{KT}\|\phi_1(0)-\phi_2(0)\|_{\Phi^w}.\end{aligned} \tag{5.128}$$

Proof:
Due to (5.110) and the fact that $\nabla_x\theta$ is uniformly bounded, we have:

$$\begin{aligned}
&\|\theta(\phi_1-\phi_2)\|^2_{L^2([0,T],H^1(\Omega))}\\
=&\|\theta(\phi_1-\phi_2)\|^2_{L^2([0,T],L^2(\Omega))}+\|\nabla_x(\theta(\phi_1-\phi_2))\|^2_{L^2([0,T],L^2(\Omega))}\\
\leqslant&\|\theta(\phi_1-\phi_2)\|^2_{L^2([0,T],L^2(\Omega))}+\|\nabla_x\theta\cdot(\phi_1-\phi_2))\|^2_{L^2([0,T],L^2(\Omega))}\\
&+\|\theta\cdot\nabla_x(\phi_1-\phi_2))\|^2_{L^2([0,T],L^2(\Omega))}\\
\leqslant&Ce^T\|\phi_1(0)-\phi_2(0)\|^2_{\Phi^w}.
\end{aligned}$$

In order to find estimates on the time derivative, we first recall that $\partial_t\phi$ verifies:

$$\partial_t\phi=(-\Delta_x+I)(\Delta_x\phi-l(t)\phi+\lambda\phi),$$

where $\phi=\phi_1-\phi_2$. Testing this equation by $\theta\varphi$ for any test function $\varphi\in C_0^\infty(\Omega)$, using that supp $\theta\subset\Omega_\delta(\phi_0)$, we obtain:

$$\begin{aligned}
(\partial_t(\theta\phi(t)),\varphi)_\Omega=&-(\Delta_x\phi(t)-l(t)\phi(t)+\lambda\phi(t),\Delta_x(\theta\varphi(t))-\theta\varphi(t))_\Omega\\
=&(\nabla_x\phi(t),\nabla_x\Delta_x(\theta\varphi(t)))_\Omega-(l(t)\phi(t),\theta\varphi(t))_\Omega+(\lambda\phi(t),\theta\varphi(t))_\Omega\\
&-(\nabla_x\phi(t),\nabla_x\theta\varphi(t))_\Omega+(l(t)\phi(t),\Delta_x(\theta\varphi(t)))_\Omega-(\lambda\phi(t),\Delta_x(\theta\varphi(t)))_\Omega\\
\leqslant&C\|\phi\|_{H^1(\Omega)}\|\varphi\|_{H^3(\Omega)}.
\end{aligned}\tag{5.129}$$

Estimate (5.129) yields that $\|\partial_t(\theta(\phi_1(t)-\phi_2(t)))\|_{H^{-3}(\Omega)}\leqslant C\|\phi\|_{H^1(\Omega)}$ and using (5.110), we obtain (5.128). □
To conclude the proof of the theorem, we introduce the functional spaces:

$$\begin{aligned}
&\mathbb{H}_1:=L^2([0,T],H^1(\Omega))\cap H^1([0,T],H^{-3}(\Omega)),\\
&\mathbb{H}:=L^2([0,T],L^2(\Omega)).
\end{aligned}$$

We have that $\mathbb{H}_1$ is compactly embedded into $\mathbb{H}$. For every $\phi_0\in\mathbb{B}_0$, we define the following operator:

$$\begin{aligned}
\mathbb{K}_{\phi_0}:B(\varepsilon,\phi_0,\Phi^w)&\longmapsto\mathbb{H}_1\\
\phi(0)&\longmapsto\mathbb{K}_{\phi_0}\phi(0):=\theta\phi(\cdot),
\end{aligned}$$

where $\phi(t)$ is the variational solution departing from $\phi(0)$. Due to Lemma 5.5.6, the map $\mathbb{K}_{\phi_0}$ is uniformly Lipschitz continuous:

$$\|\mathbb{K}_{\phi_0}(\phi_1-\phi_2)\|_{\mathbb{H}_1}\leqslant L\|\phi_1-\phi_2\|_{\Phi^w},\ \phi_1,\phi_1\in B(\varepsilon,\phi_0,\Phi^w),\ \varepsilon\leqslant\varepsilon_0\tag{5.130}$$

and thanks to Lemma 5.118, we have:

$$\|S(T)\phi_1-S(T)\phi_2\|_{\Phi^w}\leqslant\gamma\|\phi_1-\phi_2\|_{\Phi^w}+c\|\mathbb{K}_{\phi_0}(\phi_1-\phi_2)\|_{\mathbb{H}},\tag{5.131}$$

where $\gamma<0$, $c>0$ are independent of $\phi_0\in\mathbb{B}_0$, $\varepsilon\leqslant\varepsilon_0$ and $\phi_1,\phi_1\in B(\varepsilon,\phi_0;\Phi^w)$. Arguing as in Miranville & Zelik [45], we have that inequalities (5.130) and (5.131), together with the compactness of the embedding $\mathbb{H}_1\subset\mathbb{H}$, guarantee the existence of an exponential attractor $\mathcal{M}_d\subset\mathbb{B}_0$ for the discrete semigroup $S(nT)$ acting on

the phase space $\mathbb{B}_0$. Since the semigroup $S(t)$ is uniformly Hölder continuous with respect to time and space in $[0,T] \times \mathbb{B}_0$, we deduce the existence of an exponential attractor $\mathcal{M}$ for the continuous semigroup $S(t)$ on $\mathbb{B}_0$ which can be obtained by the standard formula

$$\mathcal{M} := \cup_{t\in[0,T]} \mathcal{M}_d.$$

□

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Bibliographie

[1] A. Babin and B. Nicolaenko. Exponential attractors of reaction-diffusion systems in an unbounded domain. *J. Dynam. Differential Equations*, 7(4) :567–590, 1995.

[2] A. V. Babin and M. I. Vishik. Attractors of partial differential evolution equations in an unbounded domain. *Proc. Roy. Soc. Edinburgh Sect. A*, 116(3-4) :221–243, 1990.

[3] A. Bonfoh and A. Miranville. On Cahn-Hilliard-Gurtin equations. *Nonlinear Anal.*, 47(5) :3455–3466, 2001.

[4] M. Carrive, A. Miranville, A. Piétrus, and J. M. Rakotoson. Weakly coupled dynamical systems and applications. *Asymptot. Anal.*, 30(2) :161–185, 2002.

[5] L. Cherfils, S. Gatti, and A. Miranville. Existence of global solutions to the Caginalp phase-field system with dynamic boundary conditions and singular potentials. *J. Math. Anal. Appl.*, 343(1) :557–566, 2008.

[6] L. Cherfils, S. Gatti, and A. Miranville. Long time behavior of the Caginalp system with singular potentials and dynamic boundary conditions. *Commun. Pure Appl. Anal.*, 11(6) :2261–2290, 2012.

[7] L. Cherfils, M. Petcu, and M. Pierre. A numerical analysis of the Cahn-Hilliard equation with dynamic boundary conditions. *Discrete Contin. Dyn. Syst.*, 27(4) :1511–1533, 2010.

[8] R. Chill and M. A. Jendoubi. Convergence to steady states in asymptotically autonomous semilinear evolution equations. *Nonlinear Anal.*, 53(7-8) :1017–1039, 2003.

[9] A. Eden, C. Foias, B. Nicolaenko, and R. Temam. *Exponential attractors for dissipative evolution equations*, volume 37 of *RAM : Research in Applied Mathematics*. Masson, Paris, 1994.

[10] M. Efendiev, A. Miranville, and S. Zelik. Exponential attractors for a nonlinear reaction-diffusion system in $\mathbf{R}^3$. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(8) :713–718, 2000.

[11] M. Efendiev, A. Miranville, and S. Zelik. Exponential attractors for a singularly perturbed Cahn-Hilliard system. *Math. Nachr.*, 272 :11–31, 2004.

[12] C. M. Elliott, D. A. French, and F. A. Milner. A second order splitting method for the Cahn-Hilliard equation. *Numer. Math.*, 54(5) :575–590, 1989.

[13] C.M. Elliott. The Cahn-Hilliard model for the kinetics of phase separation. In *Mathematical models for phase change problems (Óbidos, 1988)*, volume 88 of *Internat. Ser. Numer. Math.*, pages 35–73. Birkhäuser, Basel, 1989.

[14] C.M. Elliott and S. Larsson. Error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation. *Math. Comp.*, 58(198) :603–630, S33–S36, 1992.

[15] A. Ern and J.L. Guermond. *Éléments finis : théorie, applications, mise en œuvre*, volume 36 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2002.

[16] C. Gal. Well-posedness and long time behavior of the non-isothermal viscous Cahn-Hilliard equation with dynamic boundary conditions. *Dyn. Partial Differ. Equ.*, 5(1) :39–67, 2008.

[17] C. G. Gal. Global well-posedness for the non-isothermal Cahn-Hilliard equation with dynamic boundary conditions. *Adv. Differential Equations*, 12(11) :1241–1274, 2007.

[18] C. G. Gal and A. Miranville. Robust exponential attractors and convergence to equilibria for non-isothermal Cahn-Hilliard equations with dynamic boundary conditions. *Discrete Contin. Dyn. Syst. Ser. S*, 2(1) :113–147, 2009.

[19] G. Gilardi, A. Miranville, and G. Schimperna. On the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions. *Commun. Pure Appl. Anal.*, 8(3) :881–912, 2009.

[20] G. Gilardi, A. Miranville, and G. Schimperna. Long time behavior of the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions. *Chin. Ann. Math. Ser. B*, 31(5) :679–712, 2010.

[21] C. Giorgi, M. Grasselli, and V. Pata. Uniform attractors for a phase-field model with memory and quadratic nonlinearity. *Indiana Univ. Math. J.*, 48(4) :1395–1445, 1999.

[22] G. R. Goldstein and A. Miranville. A Cahn-Hilliard-Gurtin model with dynamic boundary conditions. *Discrete Contin. Dyn. Syst. Ser. S*, 6(2) :387–400, 2013.

[23] G. R. Goldstein, A. Miranville, and G. Schimperna. A Cahn-Hilliard model in a domain with non-permeable walls. *Phys. D*, 240(8) :754–766, 2011.

[24] Alain Haraux and Mohamed Ali Jendoubi. On a second order dissipative ODE in Hilbert space with an integrable source term. *Acta Math. Sci. Ser. B Engl. Ed.*, 32(1) :155–163, 2012.

[25] S. Injrou and M. Pierre. Stable discretizations of the Cahn-Hilliard-Gurtin equations. *Discrete Contin. Dyn. Syst.*, 22(4) :1065–1080, 2008.

[26] S. Injrou and M. Pierre. Error estimates for a finite element discretization of the Cahn-Hilliard-Gurtin equations. *Adv. Differential Equations*, 15(11-12) :1161–1192, 2010.

[27] H Israel. Long time behavior of an Allen-Cahn type equation with a singular potential and dynamic boundary conditions. *J. Appl. Anal. Comput.*, 2(1) :29–56, 2012.

[28] H. Israel. Well-posedness and long time behavior of an Allen-Cahn type equation. *Commun. Pure Appl. Anal.*, 12(6) :2811–2827, 2013.

[29] H. Israel, A. Miranville, and M. Petcu. Well-posedness and long time behavior of a perturbed Cahn-Hilliard system with regular potentials. *J. Asymptotic Anal., to appear*, 84 :147–179, 2013.

[30] G. Karali and M.A. Katsoulakis. The role of multiple microscopic mechanisms in cluster interface evolution. *J. Differential Equations*, 235(2) :418–438, 2007.

[31] G. Karali and T. Ricciardi. On the convergence of a fourth order evolution equation to the Allen-Cahn equation. *Nonlinear Anal.*, 72(11) :4271–4281, 2010.

[32] M. A. Katsoulakis and D. G. Vlachos. From microscopic interactions to macroscopic laws of cluster evolution. *Phys. Rev. Lett.*, 84 :1511–1514, 2000.

[33] R. Kenzler, F. Eurich, P. Maass, B. Rinn, J. Schropp, E. Bohl, and W. Dieterich. Phase separation in confined geometries : solving the Cahn-Hilliard equation with generic boundary conditions. *Comput. Phys. Comm.*, 133(2-3) :139–157, 2001.

[34] Mihály Kovács, Stig Larsson, and Ali Mesforush. Finite element approximation of the Cahn-Hilliard-Cook equation. *SIAM J. Numer. Anal.*, 49(6) :2407–2429, 2011.

[35] Stig Larsson and Ali Mesforush. Finite-element approximation of the linearized Cahn-Hilliard-Cook equation. *IMA J. Numer. Anal.*, 31(4) :1315–1333, 2011.

[36] B. Merlet and M. Pierre. Convergence to equilibrium for the backward Euler scheme and applications. *Commun. Pure Appl. Anal.*, 9(3) :685–702, 2010.

[37] A. S. Mikhailov, M. Hildebrand, and G. Ertl. Nonequilibrium nanostructures in condensed reactive systems. In *Coherent structures in complex systems (Sitges, 2000)*, volume 567 of *Lecture Notes in Phys.*, pages 252–269. Springer, Berlin, 2001.

[38] A. Miranville. Some generalizations of the Cahn-Hilliard equation. *Asymptot. Anal.*, 22(3-4) :235–259, 2000.

[39] A. Miranville. Long-time behavior of some models of Cahn-Hilliard equations in deformable continua. *Nonlinear Anal. Real World Appl.*, 2(3) :273–304, 2001.

[40] A. Miranville, A. Piétrus, and J. M. Rakotoson. Dynamical aspect of a generalized Cahn-Hilliard equation based on a microforce balance. *Asymptot. Anal.*, 16(3-4) :315–345, 1998.

[41] A. Miranville and S. Zelik. Robust exponential attractors for Cahn-Hilliard type equations with singular potentials. *Math. Methods Appl. Sci.*, 27(5) :545–582, 2004.

[42] A. Miranville and S. Zelik. Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions. *Math. Methods Appl. Sci.*, 28(6) :709–735, 2005.

[43] A. Miranville and S. Zelik. Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions. *Math. Methods Appl. Sci.*, 28(6) :709–735, 2005.

[44] A. Miranville and S. Zelik. Attractors for dissipative partial differential equations in bounded and unbounded domains. In *Handbook of differential equations : evolutionary equations. Vol. IV*, Handb. Differ. Equ., pages 103–200. Elsevier/North-Holland, Amsterdam, 2008.

[45] A. Miranville and S. Zelik. The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions. *Discrete Contin. Dyn. Syst.*, 28(1) :275–310, 2010.

[46] Alain Miranville and Arnaud Rougirel. Local and asymptotic analysis of the flow generated by the Cahn-Hilliard-Gurtin equations. *Z. Angew. Math. Phys.*, 57(2) :244–268, 2006.

[47] A. Novick-Cohen. The Cahn-Hilliard equation. In *Handbook of differential equations : evolutionary equations. Vol. IV*, Handb. Differ. Equ., pages 201–228. Elsevier/North-Holland, Amsterdam, 2008.

[48] J. Prüss, R. Racke, and S. Zheng. Maximal regularity and asymptotic behavior of solutions for the Cahn-Hilliard equation with dynamic boundary conditions. *Ann. Mat. Pura Appl. (4)*, 185(4) :627–648, 2006.

[49] R. Racke and S. Zheng. The Cahn-Hilliard equation with dynamic boundary conditions. *Adv. Differential Equations*, 8(1) :83–110, 2003.

[50] J. C. Robinson. *Infinite-dimensional dynamical systems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001. An introduction to dissipative parabolic PDEs and the theory of global attractors.

[51] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.

[52] Batoul Saoud. Pullback exponential attractors for the viscous Cahn-Hilliard equation in bounded domains. *J. Appl. Anal. Comput.*, 2(1) :73–90, 2012.

[53] J. Shen and X. Yang. Numerical approximations of Allen-Cahn and Cahn-Hilliard equations. *Discrete Contin. Dyn. Syst.*, 28(4) :1669–1691, 2010.

[54] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1997.

[55] V. Thomée. *Galerkin finite element methods for parabolic problems*, volume 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2006.

[56] H. Wu and S. Zheng. Convergence to equilibrium for the Cahn-Hilliard equation with dynamic boundary conditions. *J. Differential Equations*, 204(2) :511–531, 2004.

[57] J. Zhang and Q. Du. Numerical studies of discrete approximations to the Allen-Cahn equation in the sharp interface limit. *SIAM J. Sci. Comput.*, 31(4) :3042–3063, 2009.

Comportement asymptotique de modèles en séparation de phase

Résumé : Dans cette thèse, on étudie l'existence, l'unicité et la régularité des solutions d'équation de type Cahn-Hilliard ainsi que son comportement asymptotique en termes d'existence de l'attracteur global et d'un attracteur exponentiel. Cette équation est considérée dans un domaine borné et régulier pour différents types de nonlinéarités et de conditions au bord.
D'abord, on étudie l'équation avec des conditions de type Dirichlet sur le bord et une nonlinéarité régulière. Après, on considère une perturbation du problème et on démontre l'existence d'une famille robuste d'attracteurs exponentiels lorsque ε tend vers 0.
Ensuite, on étudie l'équation avec des conditions dynamiques sur le bord. On considère tout d'abord une nonlinéarité régulière et on donne une étude théorique et numérique. Après, on illustre ces résultats par des simulations numériques en dimension deux d'espace qui permettent d'étudier l'influence des différents paramètres. On termine par une étude du modèle considéré avec une nonlinéarité singulière que l'on approche par des fonctions régulières et on introduit une notion de solution appropriée.

Mot-clefs : Equation de Cahn-Hilliard, comportement asymtotique des solutions, attracteur global, attracteur exponentiel, analyse numérique.

Asymptotic Behaviour of Some Phase Separation Models

Abstract : This thesis is devoted to the study of the existence, uniqueness and regularity of solutions for a Cahn-Hilliard type equation, as well as the asymptotic behavior in terms of existence of the global attractor and of an exponential attractor. This equation is considered in a bounded and smooth domain under various assumptions on the nonlinear terms and with different boundary conditions.
We start by studying the equation with Dirichlet boundary conditions and a regular nonlinearity. Then, we consider a perturbation of the problem and we prove the existence of a robust family of exponential attractors as ε tends to 0.
For the equation endowed with dynamic boundary conditions, we first consider a regular nonlinearity and we treat the theoretical and numerical analysis. Then, we illustrate the results by numerical simulations in two space dimension which allow us to study the influence of different parameters. Finally, we treat the problem considered with a singular nonlinearity which is approximated by regular functions and we give a suitable notion of solutions.

Keywords : Cahn-Hilliard Equation, well posedness, dissipativity, asymptotic behavior of solutions, global attractor, exponential attractor, numerical analysis.