THÈSE

Pour l'obtention du grade de
DOCTEUR DE L'UNIVERSITÉ DE POITIERS
UFR des sciences fondamentales et appliquées
Laboratoire de mathématiques et applications - LMA (Poitiers)
(Diplôme National - Arrêté du 7 août 2006)

École doctorale : Sciences et ingénierie pour l'information, mathématiques - S2IM (Poitiers)
Secteur de recherche : Mathématiques et leurs applications

Présentée par :
Frédéric Eyi-Minko

Propriétés des processus max-stables : théorèmes limites, lois conditionnelles et mélange fort

Directeur(s) de Thèse :
Clément Dombry

Soutenue le 11 octobre 2013 devant le jury

Jury :

Président	**Julien Michel**	Professeur des Universités, Université de Poitiers
Rapporteur	**Philippe Soulier**	Professeur des Universités, Université Paris Ouest Nanterre
Rapporteur	**Stilian Stoev**	Professor, University of Michigan, USA
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Membre	**Hermine Biermé**	Maître de conférences, Université de Paris Descartes

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Hermine BIERMÉ	Université Paris Descartes	Examinateur
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Introduction

1 Contexte et objectifs

La théorie des valeurs extrêmes est une théorie relativement récente, les premières publications dessus ne datent que de 1927 (Fréchet [41]) et 1928 (Fisher and Tippett [40]). Elle possède néanmoins plusieurs domaines d'applications, tels que par exemple l'environnement ou la finance, dont à chaque fois la particularité du problème traité est le fait que l'on s'intéresse aux maximums ou aux minimums de données collectées, ou plus généralements aux valeurs extrêmes. Cette thèse se place dans le domaine de la théorie spatiale des valeurs extrêmes et est principalement axée sur les processus stochastiques max-stables.

Définition 1.1. *On dit qu'un processus stochastique η est max-stable si pour tout $n \in \mathbb{N}^*$, il existe des fonctions $a_n > 0$ et b_n tels que :*

$$\max_{1 \leq i \leq n} \frac{\eta_i - b_n}{a_n} \stackrel{\mathcal{L}}{=} \eta$$

où $(\eta_i)_{1 \leq i \leq n}$ est une suite i.i.d de même loi que η et $\stackrel{\mathcal{L}}{=}$ désigne l'égalité en loi.

Ainsi, nous nous sommes intéressés à diverses propriétés des processus stochastiques max-stables : la convergence des extrêmes, les lois et les simulations conditionelles, le mélange fort, et la propriété de Markov. Cela a été le plus souvent fait à travers le prisme d'un outil qui sera utilié de façon récurrente dans cette thèse : les processus ponctuels de Poisson. En effet, d'un côté les processus stochastiques max-stables jouent un rôle essentiel dans la théorie des valeurs extrêmes, et d'un autre côté il existe une forte liaison entre les processus max-stables et les processus ponctuels de Poisson mis par exemple en évidence dans une publication de Giné et al. [44]. Ainsi, des auteurs comme Smith [70] ou encore de Haan [30] ont donné des méthodes de constructions de processus max-stables à partir de processus ponctuels de Poisson abondamment utilisés dans la littérature.

2 Extrêmes de champs aléatoires i.i.d. : une approche par les processus ponctuels

Comme l'indique son titre, le Chapitre 2 de cette thèse traite des processus de maxima partiels, et cela, à travers l'étude de processus ponctuels. Soit (E, d) un espace métrique complet séparable et $\mathcal{E}$ la tribu borélienne de E. Une mesure borélienne μ sur E est dite finie sur les bornés si, pour tout borélien borné B, $\mu(B) < \infty$. On notera $M_b^\sharp(E)$ l'espace des mesures boréliennes finies sur les bornés. Pour $\{\mu_n, n \geqslant 1\}$ et μ des mesures de $M_b^\sharp(E)$, on dira que $\{\mu_n, n \geqslant 1\}$ converge $\sharp$-faiblement vers μ quand $n \to \infty$, et on notera alors $\mu_n \overset{w^\sharp}{\to} \mu$, si $\int f d\mu_n \to \int f d\mu$ quand $n \to \infty$ pour toute fonction f bornée et continue de (E, d) à support borné. Muni de cette notion de convergence, $M_b^\sharp(E)$ est un espace métrisable complet séparable.

Soit T un espace métrique complet, on notera $\mathbb{C}^+ = \mathbb{C}^+(T)$ l'espace des fonctions positives et continues sur T muni de la topologie de la convergence uniforme sur les compacts. Pour $n \in \mathbb{N}$, soit $(X_{i,n})_{i \geqslant 1}$ est suite de processus i.i.d de même loi qu'un certain processus $(X_n(t))_{t \in T}$ à trajectoire continue que nous supposerons positif ($E = [0, \infty)$). L'un des principaux résultats du Chapitre 2 est la convergence des deux processus suivants :

$$M_n(t) = \max\{X_{in}(t),\ 1 \leqslant i \leqslant n\}, \quad t \in T,$$

et

$$\tilde{M}_n(u,t) = \max\{X_{in}(t),\ 1 \leqslant i \leqslant [nu]\}, \quad u \geq 0,\ t \in T.$$

On utilise la convention $\tilde{M}_n(0,t) \equiv 0$. La convergence du processus M_n ne sera pas directement établie sur $\mathbb{C}^+$, mais sur l'ensemble $\overline{\mathbb{C}}_0^+ = (0, +\infty] \times \mathbb{S}_{\mathbb{C}^+}$ où $\mathbb{S}_{\mathbb{C}^+} = \{x \in \mathbb{C}^+ : \|x\|_\infty = 1\}$ est la sphère unitaire, munie d'une métrique plus adaptée aux outils manipulés pour la démonstration :

$$d((r_1, s_1), (r_2, s_2)) = |1/r_1 - 1/r_2| + \|s_1 - s_2\|, \quad (r_1, s_1), (r_2, s_2) \in \overline{\mathbb{C}}_0^+.$$

Cette métrique a l'avantage de rendre les ensembles de la forme $(\varepsilon, \infty] \times \mathbb{S}_{\mathbb{C}^+}$ bornés. Ainsi $(\overline{\mathbb{C}}_0^+, d)$ est un espace complet et séparable et les ensembles $(0, \infty) \times \mathbb{S}_{\mathbb{C}^+}$ et $\mathbb{C}_0^+ = \mathbb{C}^+ \setminus \{0\}$ sont homéomorphes. Ceci nous permet d'obtenir le théorème suivant.

Théorème 2.1. *Supposons que $n\mathbb{P}[X_n \in \cdot\,] \overset{w^\sharp}{\to} \nu$ dans $M_b^\sharp(\overline{\mathbb{C}}_0^+)$ et que $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$. Alors, le processus M_n converge faiblement dans $\mathbb{C}^+$ quand $n \to \infty$ vers le processus M défini par*

$$M(t) = \sup_{i \geq 1} Y_i(t), \quad t \in T,$$

où $\sum_{i \geq 1} \delta_{Y_i}$ est un processus ponctuel de Poisson sur $\mathbb{C}_0^+$ d'intensité ν.

Considérons maintenant la convergence de $\tilde{M}_n$. Pour $u \geq 0$ fixé, on a $\tilde{M}_n(u, \cdot) \in \mathbb{C}^+$, par contre la fonction $u \mapsto \tilde{M}_n(u, \cdot)$ à valeur dans $\mathbb{C}^+$ est càd-làg sur $[0, +\infty)$. Ainsi, nous considérerons le processus $\tilde{M}_n$ comme un élément aléatoire de l'espace de Skohorod $\mathbb{D}([0, +\infty), \mathbb{C}^+)$ muni de topologie J_1 (pour la définition et les propriétés de l'espace de Skohorod voir par exemple Ethier et Kurtz [39]).

Théorème 2.2. *Supposons que $n\mathbb{P}[X_n \in \cdot\,] \overset{w^\sharp}{\to} \nu$ dans $M_b^\sharp(\overline{\mathbb{C}}_0^+)$ et que $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$. Le processus $\tilde{M}_n$ converge alors faiblement dans $\mathbb{D}([0, +\infty), \mathbb{C}^+)$ quand $n \to \infty$ vers le processus super-extrémal $\tilde{M}$ défini par*

$$\tilde{M}(u,t) = \sup\{Y_i(t) \mathbf{1}_{[U_i, +\infty)}(u);\ i \geq 1\}, \quad u > 0,\ t \in T,$$

où $\sum_{i \geq 1} \delta_{(Y_i, U_i)}$ est un processus ponctuel de Poisson sur $\mathbb{C}_0^+ \times [0, +\infty)$ d'intensité $\nu \otimes \ell$, ℓ désignant la mesure de Lesbegue sur $[0, \infty)$.

Par la suite nous nous sommes intéressés à diverses propriétés du processus limite $\tilde{M}$. Par exemple, nous avons déterminé ses lois fini-dimensionnelles, en utilisant les notations suivantes : $l \geq 1$, $\mathbf{t} = (t_1, \dots, t_l) \in T^l$, $\mathbf{y} = (y_1, \dots, y_l) \in [0, +\infty)^l$, et pour $u > 0$, nous avons écrit $\tilde{M}(u, \mathbf{t}) \leq \mathbf{y}$ pour dire que $\tilde{M}(u, t_i) \leq y_i$ pour tout $i \in \{1, \dots, l\}$.

Proposition 2.1. *Pour $k \geqslant 1$, $0 = u_0 < u_1 < \cdots < u_k$, $\mathbf{t} \in T^l$ et $\mathbf{y}_1, \dots, \mathbf{y}_k \in [0, +\infty)^l$, on a*

$$\mathbb{P}\Big[\tilde{M}(u_1, \mathbf{t}) \leqslant \mathbf{y}_1, ..., \tilde{M}(\mathbf{u}_k, \mathbf{t}) \leqslant \mathbf{y}_k\Big] = \prod_{i=1}^{k} \exp[-(u_j - u_{j-1})\nu(A_j)]$$

avec $A_j = \{f \in \mathbb{C}_0^+;\ \exists i \in 1, \dots, l,\ f(t_i) \geq \min_{k \geq j} y_{k,i}\}$.

Des propriétés reposant sur l'homogénéité de la mesure ν ont aussi été données, rappelons que l'on dit que ν est homogène d'indice $-\alpha < 0$ lorsque pour tout $u \geq 0$, $\nu(uA) = u^\alpha \nu(A)$.

Proposition 2.2. *Supposons que ν soit homogène d'indice $-\alpha < 0$, alors :*

- *$\tilde{M}$ est max-stable d'indice $\alpha > 0$, i.e. si $\tilde{M}^1, \dots, \tilde{M}^n$ sont i.i.d de même loi que $\tilde{M}$, le maximum $\vee_{i=1}^n \tilde{M}^i$ a la même loi que $n^{\frac{1}{\alpha}} \tilde{M}$;*
- *$\tilde{M}$ est auto-similaire d'indice $\frac{1}{\alpha}$, i.e. pour tout $c > 0$, le processus $(\tilde{M}(cu, \cdot))_{u \geq 0}$ possède la même loi que $(c^{1/\alpha} \tilde{M}(cu, \cdot))_{u \geq 0}$.*

Des propriétés sur les statistiques d'ordres ont aussi été établies. Pour $1 \leq r \leq n$ et $t \in T$, on note $M_n^r(t)$ le $r^{\text{ème}}$ plus grand élément parmi $X_{1n}(t), \dots, X_{nn}(t)$, aussi $M_n^r = (M_n^r(t))_{t \in T}$ est appelé la statistique d'ordre r de $\{X_{1n}, \dots, X_{1n}\}$. De la même façon, pour $r \geq 1$, $t \in T$ et $u \geq 0$, on définit $\tilde{M}_n^r(u, t)$ comme étant le $r^{\text{ème}}$ plus grand élément parmi $X_{1n}(t), \dots, X_{[nu]n}(t)$. Dans les deux cas si $r > n$, alors la statistique d'ordre r est considérée comme nulle. Enfin, considérons l'ensemble $B_{u,t}^y = \{(f, v) \in \mathbb{C}_0^+ \times [0, +\infty); f(t) \geq y, v \leq u\}$. Nous avons la propriété suivante.

Théorème 2.3. *Supposons que $n\mathbb{P}[X_n \in \cdot\,] \xrightarrow{w^\sharp} \nu$ dans $M_b^\sharp(\overline{\mathbb{C}}_0^+)$ et que $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$. Alors pour tout $r \geq 1$, le vecteur joint de statistiques d'ordre $(\tilde{M}_n^1, \dots, \tilde{M}_n^r)$ converge faiblement dans $\mathbb{D}([0, +\infty), (\mathbb{C}^+)^r)$ quand $n \to \infty$ vers $(\tilde{M}^1, \dots, \tilde{M}^r)$ défini par*

$$\tilde{M}^j(u, t) = \sup\{y \geq 0;\ \tilde{\Pi}_\nu \cap B_{u,t}^y \geq r\}, \quad 1 \leq j \leq r,\ u \geq 0,\ t \in T,$$

avec $\tilde{\Pi}_\nu$ un processus ponctuel de Poisson sur $\mathbb{C}_0^+ \times [0, +\infty)$ d'intensité $\nu \otimes \ell$.

Le Chapitre 2 a été conclu par l'illustration de différents résultats dans le cas des processus gaussiens.

3 Lois conditionnelles des processus max-infiniment divisibles et simulations conditionnelles des processus max-stables

3.1 Lois conditionnelles des processus max-infiniment divisibles

Soient T un espace métrique compact et $\mathbb{C}_0$ l'espace des fonctions continues positives et non nulles sur T muni de la norme sup. Le Chapitre 3 est consacré à l'étude des lois conditionnelles des processus max-infiniment divisibles (max-i.d.).

Définition 3.1. *On dit que $\eta = \{\eta(t)\}_{t\in T}$, un processus stochastique de $\mathbb{C}_0$, est max-infiniment divisible (max-i.d.) si pour tout $n \geq 1$, il existe des processus $\{\eta_{ni},\ 1 \leq i \leq n\}$ de $\mathbb{C}_0$ i.i.d. tels que :*

$$\eta \stackrel{\mathcal{L}}{=} \bigvee_{i=1}^{n} \eta_{ni},$$

où $\bigvee$ désigne le maximum ponctuel.

Considérons $\eta = \{\eta(t)\}_{t\in T}$ un processus stochastique max-i.d. de $\mathbb{C}_0$. Soit $M_p(\mathbb{C}_0)$ l'espace des mesures ponctuelles sur $\mathbb{C}_0$ finies sur tous les ensembles de la forme $\{f_i \in \mathbb{C}_0 \colon \|f_i\| > \varepsilon\}$, $\forall\varepsilon > 0$. D'après le Théoreme 2.4 de [44], on sait qu'il existe une mesure μ sur $\mathbb{C}_0$, qui est la mesure d'intensité de $\Phi = \sum_{i=1}^{N} \delta_{\phi_i}$ un processus ponctuel de Poisson de $M_p(\mathbb{C}_0)$ tel que :

$$\eta(t) \stackrel{\mathcal{L}}{=} \max\{\phi(t); f \in [\Phi]\}, \quad t \in T,$$

où $[\Phi]$ désigne l'ensemble des atomes de $\Phi \in M_p(\mathbb{C}_0)$. La mesure μ est alors appelée la mesure exponentielle de η. Pour le vecteur $\mathbf{s} = (s_1, \dots, s_l) \in T^l$ et toute fonction $f \in \mathbb{C}_0$, nous noterons $f(\mathbf{s}) = (f(s_1), \dots, f(s_l))$, et $\mu_\mathbf{s}$ la mesure exponentielle du vecteur aléatoire max-i.d. $\eta(\mathbf{s})$ définie sur $[0, +\infty)^l \setminus \{0\}$ par :

$$\mu_\mathbf{s}(A) = \mu(\{f \in \mathbb{C}_0 :\ f(\mathbf{s}) \in A\}), \quad A \subset [0, +\infty)^l \setminus \{0\} \text{ borélien.}$$

Toujours pour le vecteur $\mathbf{s} = (s_1, \dots, s_l) \in T^l$, définissons aussi la fonction suivante :

$$\bar{\mu}_\mathbf{s}(\mathbf{x}) = \mu(\{f \in \mathbb{C}_0 : \exists s_i \in \{s_1, \dots, s_l\},\ f(s_i) \geq x_i \text{ et } f(\mathbf{s}) \neq 0\}), \quad \mathbf{x} = (x_1, \dots, x_l) \in [0, +\infty)^l.$$

La proposition suivante nous permettra de définir un outil que nous utiliserons souvent : les fonctions extrémales.

Proposition 3.1. *Pour tout $t \in T$, les assertions suivantes sont équivalentes :*

(i) *Il n'existe presque sûrement qu'un unique un atome de Φ noté ϕ_t^+ tel que $\eta(t) = \phi_t^+(t)$;*

(ii) *$\bar{\mu}_t(0^+) = +\infty$ et $\bar{\mu}_t$ est continue sur $(0, +\infty)$;*

(iii) *la loi de $\eta(t)$ n'a pas d'atomes.*

Dans le cas où les assertions de la proposition précédente sont vérifiées, ϕ_t^+ sera appelée la fonction extrémale en t. Nous allons travailler avec $K = \{t_1, \dots, t_k\}$ un ensemble fini de T, tel que pour tout $t \in K$, la fonction $\bar{\mu}_t$ soit continue et $\bar{\mu}_t(0^+) = +\infty$. Par la suite nous noterons $\mathbf{t} = (t_1, \dots, t_k)$ et $\mathbf{y} = (y_1, \dots, y_k) \in [0, +\infty)^k$, aussi pour tout ensemble non vide $L \subset K$, nous adopterons les notations suivantes : $\tilde{L} = \{i \in [\![1, k]\!] : t_i \in L\}$, $\mathbf{t}_L = (t_i)_{i\in\tilde{L}}$, $\mathbf{y}_L = (y_i)_{i\in\tilde{L}}$ et $L^c = K \smallsetminus L$.

Définition 3.2. *Soit $\sim$ la relation d'équivalence (aléatoire) sur $K = \{t_1, \dots, t_k\}$ définie par $t \sim t'$ si et seulement si $\phi_t^+ = \phi_{t'}^+$. La partition $\Theta = (\theta_1, \dots, \theta_{\ell(\Theta)})$ de K en classes d'équivalences sera appelée hitting scénario. Pour $j \in [\![1, \ell(\Theta)]\!]$, nous noterons φ_j^+ la fonction extrémale associée aux éléments θ_j, i.e., telle que $\varphi_j^+ = \phi_t^+$ pour tout $t \in \theta_j$.*

Pour $\mathbf{x} \in [0,+\infty)^l \setminus \{0\}$, soit $P_{\mathbf{s}}(\mathbf{x}, df)$ la loi conditionnelle de $\mu(df)$ sachant $f(\mathbf{s}) = \mathbf{x}$ (plus d'amples explications seront données sur cette mesure conditionelle dans l'Appendix 3.A.2), c'est à dire que pour toute fonction mesurable

$$F : [0,+\infty)^l \times \mathbb{C}_0 \to [0,+\infty)$$

s'annulant sur $\{0\} \times \mathbb{C}_0$, nous avons :

$$\int_{\mathcal{C}_0} F(f(\mathbf{s}), f)\mu(df) = \int_{[0,+\infty)^l \setminus \{0\}} \int_{\mathcal{C}_0} F(\mathbf{x}, f) P_{\mathbf{s}}(\mathbf{x}, df)\mu_{\mathbf{s}}(d\mathbf{x}).$$

Soit $\mathcal{P}_K$ l'ensemble des partitions de K. Pour $\tau \in \mathcal{P}_K$ de cardinal ℓ, nous noterons $\nu_{\mathbf{t}}^{\tau}$ la mesure sur $[0,+\infty)^k$ définie par :

$$\nu_{\mathbf{t}}^{\tau}(C) = \mathbb{P}(\eta(\mathbf{t}) \in C;\ \Theta = \tau),$$

où $C \subset [0,+\infty)^k$ est un borélien. Ainsi $\nu_{\mathbf{t}}$ la loi de $\eta(\mathbf{t})$ est donnée par $\nu_{\mathbf{t}} = \sum_{\tau \in \mathcal{P}_K} \nu_{\mathbf{t}}^{\tau}$. Maintenant considérons les lois conditionelles suivantes :

$$\begin{aligned}
\pi_{\mathbf{t}}(\mathbf{y}, \cdot) &= \mathbb{P}[\Theta \in \cdot \mid \eta(\mathbf{t}) = \mathbf{y}] \\
Q_{\mathbf{t}}(\mathbf{y}, \tau, \cdot) &= \mathbb{P}[(\varphi_j^+)_{1 \le j \le \ell} \in \cdot \mid \eta(\mathbf{t}) = \mathbf{y}, \Theta = \tau] \\
R_{\mathbf{t}}(\mathbf{y}, \tau, (f_j)_{1 \le j \le \ell}, \cdot) &= \mathbb{P}[\Phi_K^- \in \cdot \mid \eta(\mathbf{t}) = \mathbf{y}, \Theta = \tau, (\varphi_j^+)_{1 \le j \le \ell} = (f_j)_{1 \le j \le \ell}],
\end{aligned}$$

où $\Phi_K^- = \Phi - \sum_{j=1}^{\ell(\Theta)} \varphi_j^+$. Nous avons pu déterminer des expressions explicites de ces lois, le résultat est dans le théorème suivant :

Théorème 3.3. *Nous avons les trois résultats suivants :*

1. *Pour tout $\tau \in \mathcal{P}_K$, nous avons $\nu_{\mathbf{t}}(d\mathbf{y})$-presque partout :*

$$\pi_{\mathbf{t}}(\mathbf{y}, \tau) = \frac{d\nu_{\mathbf{t}}^{\tau}}{d\nu_{\mathbf{t}}}(\mathbf{y})$$

où $d\nu_{\mathbf{t}}^{\tau}/d\nu_{\mathbf{t}}$ est la dérivée de Radon-Nykodym de $\nu_{\mathbf{t}}^{\tau}$ par rapport à $\nu_{\mathbf{t}}$.

2. *Nous avons $\nu_{\mathbf{t}}(d\mathbf{y})\pi_{\mathbf{t}}(\mathbf{y}, d\tau)$-presque partout :*

$$Q_{\mathbf{t}}(\mathbf{y}, \tau, df_1 \cdots df_\ell) = \bigotimes_{j=1}^{\ell} \left\{ \frac{1_{\{f_j(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\}} P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)}{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\})} \right\}.$$

3. *Soit $C_{\mathbf{t}}^-(\mathbf{y}) = \{M \in M_p(\mathbb{C}_0);\ \forall f \in [M],\ f(\mathbf{t}) < \mathbf{y}\}$. Pour tout ensemble mesurable $B \in M_p(\mathbb{C}_0)$, nous avons $\nu_{\mathbf{t}}(d\mathbf{y})$-presque partout :*

$$R_{\mathbf{t}}(\mathbf{y}, \tau, (f_j), B) \equiv R_{\mathbf{t}}(\mathbf{y}, B) = \frac{\mathbb{P}[\Phi \in B \cap C_{\mathbf{t}}^-(\mathbf{y})]}{\mathbb{P}[\Phi \in C_{\mathbf{t}}^-(\mathbf{y})]}.$$

Le résultat du théorème précedent nous permet d'obtenir le théorème suivant similaire au résultat déjà obtenu par Weintraub [80] dans le cas bivarié.

Théorème 3.4. *Nous avons $\nu_{\mathbf{t}}(d\mathbf{y})$-presque partout :*

$$\begin{aligned}&\mathbb{P}[\eta(\mathbf{s}) < \mathbf{z} \mid \eta(\mathbf{t}) = \mathbf{y}]\\ = \quad & \exp[-\mu(\{f(\mathbf{s}) \not< \mathbf{z},\ f(\mathbf{t}) < \mathbf{y}\})] \sum_{\tau \in \mathcal{P}_K} \pi_{\mathbf{t}}(\mathbf{y}, \tau) \prod_{j=1}^{\ell} \frac{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c},\ \ f(\mathbf{s}) < \mathbf{z}\})}{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\})}\end{aligned}$$

pour tout $l \geq 1$, $\mathbf{s} \in T^l$ et $\mathbf{z} \in [0, +\infty)^l$.

3.2 Simulations conditionnelles des processus max-stables

Le Chapitre 4 est issu d'une collaboration entre Mathieux Ribatet, Clément Dombry et moi. Dans ce chapitre, les résultats obtenus dans le Chapitre 3 ont été utilisés pour faire de la simulation. Nous nous sommes placés dans le cas où $T = \mathcal{X} \subset \mathbb{R}^d$, de plus pour tout $i \geq 1$ chaque atome du procesus ponctuel Φ est donné par :

$$\phi_i = \zeta_i Y_i$$

où $\{\zeta_i\}_{i \geq 1}$ sont les atomes d'un processus ponctuel de Poisson sur $(0, \infty)$ d'intensité $\zeta^{-2} \mathrm{d}\zeta$ et $\{Y_i\}_{i \geq 1}$ est une suite de processus stochastiques i.i.d de même loi qu'un processus stochastique postif Y tel que $E\{Y(t)\} = 1$ pour tout $t \in \mathbb{R}^d$. Les simulations du Chapitre 4 ont été faites sous la condition que Φ soit régulier, c'est à dire qu'il existe une fonction $\lambda_{\mathbf{s}}$, telle que $\mu_{\mathbf{s}}(\mathrm{d}z) = \lambda_{\mathbf{s}}(z)\mathrm{d}z$. Cette fonction est appelée fonction d'intensité. On définit également la fonction d'intensité conditionnelle :

$$\lambda_{s|t,z}(u) = \frac{\lambda_{(s,t)}(u, z)}{\lambda_t(z)}, \qquad (s, t) \in \mathcal{X}^{m+k}, \qquad u \in \mathbb{R}^m, \qquad z \in (0, +\infty)^k.$$

Ainsi dans le Chapitre 4 les simulations ont été effectuées avec deux processus max-stables très utilisés. Le processus de Brown–Resnick [16, 51] qui correspond au cas où $\forall t \in \mathbb{R}^d$, on a : $Y(t) = \exp\{W(t) - \gamma(t)\}$, W étant un processus gaussien centré à accroissement stationnaire, de variogramme γ et tel que $W(0) = 0$ presque sûrement. Le processus de Schlather [67] qui correspond au cas où $\forall t \in \mathbb{R}^d$, $Y(t) = (2\pi)^{1/2} \max\{0, \varepsilon(t)\}$ avec ε un processus gaussien standard dont on notera par ρ la fonction de correlation. Néanmoins pour que Φ soit régulier nous considérerons plutôt la représentation équivalente avec $Y(t) = (2\pi)^{1/2}\varepsilon(t)$ $(t \in \mathbb{R}^d)$. La procédure de la simulation est la suivante :

Théorème 3.5. *Soient $(x, s) \in \mathcal{X}^{k+m}$, $K = \{x_1, \ldots, x_k\}$, et $\tau = (\tau_1, \ldots, \tau_\ell) \in \mathcal{P}_K$. Considérons la procédure suivante :*

étape 1 : *créer une partition aléatoire $\theta \in \mathcal{P}_K$ de loi :*

$$\pi_x(z, \tau) = \mathrm{pr}\,\{\theta = \tau \mid Z(x) = z\} = \frac{1}{C(x, z)} \prod_{j=1}^{|\tau|} \lambda_{x_{\tau_j}}(z_{\tau_j}) \int_{\{u_j < z_{\tau_j^c}\}} \lambda_{x_{\tau_j^c} | x_{\tau_j}, z_{\tau_j}}(u_j) du_j,$$

où la constante de normalisation est

$$C(x,z) = \sum_{\tilde{\tau}\in\mathcal{P}_K} \prod_{j=1}^{|\tilde{\tau}|} \lambda_{x_{\tilde{\tau}_j}}(z_{\theta_j}) \int_{\{u_j < z_{\tilde{\tau}_j^c}\}} \lambda_{x_{\tilde{\tau}_j^c}|x_{\tilde{\tau}_j}, z_{\tilde{\tau}_j}}(u_j) du_j.$$

étape 2 : *Pour un $\tau = (\tau_1, \ldots, \tau_\ell)$ donné, créer ℓ vecteurs aléatoires indépendants $\varphi_1^+(s), \ldots, \varphi_\ell^+(s)$ à partir de la loi suivante :*

$$\mathrm{pr}\left\{\varphi_j^+(s) \in dv \mid Z(x) = z, \theta = \tau\right\} = \frac{1}{C_j}\left\{\int 1_{\{u<z_{\tau_j^c}\}} \lambda_{(s,x_{\tau_j^c})|x_{\tau_j}, z_{\tau_j}}(v,u)du\right\} dv$$

où $1_{\{\cdot\}}$ est la fonction indicatrice et

$$C_j = \int 1_{\{u<z_{\tau_j^c}\}} \lambda_{(s,x_{\tau_j^c})|x_{\tau_j}, z_{\tau_j}}(v,u)dudv.$$

On définit le vecteur aléatoire suivant $Z^+(s) = \max_{j=1,\ldots,\ell} \varphi_j^+(s)$.

étape 3 : *Indépendamment créer un processus ponctuel de Poisson $\{\zeta_i\}_{i\geq 1}$ sur $(0,\infty)$ d'intensité $\zeta^{-2}d\zeta$ et $\{Y_i\}_{i\geq 1}$ une suite i.i.d de processus de même loi qu'un processus Y, et définir le vecteur aléatoire*

$$Z^-(s) = \max_{i\geq 1} \zeta_i Y_i(s) 1_{\{\zeta_i Y_i(x) < z\}}.$$

Alors le vecteur aléatoire $\tilde{Z}(s) = \max\{Z^+(s), Z^-(s)\}$ suit la loi de $Z(s)$ sachant $Z(x) = z$.

Malheureusement même lorsque la valeur de k se trouve relativement modérée, la loi de $\pi_t(z,\cdot)$ est impossible à simuler car l'espace d'état $\mathcal{P}_K$ devient vite très large. Il a été utilisé alors pour la simulation un échantillonneur de Gibbs. Nous avons appliqué ces algorithmes de simulations conditionnelles à des données de températures et de pluies en Suisse.

4 Propriétés de mélange fort des processus max-infiniment divisibles

Nous nous sommes intéressés dans le Chapitre 5 aux coefficients de mélange, le but étant d'obtenir le théorème de la limite centrale pour des fonctionnelles additives de processus max-i.d. stationnaires. Soit T un espace métrique localement compact. Considérons $\mathbb{C}(T) = \mathbb{C}(T, [0, +\infty))$ l'espace des fonctions continues positives muni de la topologie engendrée par la convergence uniforme sur les ensembles compacts et de $\mathcal{C}$ sa tribu borélienne. Soit Φ un processus ponctuel de Poisson sur $\mathbb{C}_0(T)$ de mesure d'intensité μ, une mesure borélienne localement finie sur $\mathbb{C}_0(T) = \mathbb{C}(T) \setminus \{0\}$ et vérifiant

$$\mu\big[\{f \in \mathbb{C}_0(T);\ \sup_K f > \varepsilon\}\big] < \infty, \quad \text{pour tout compact } K \subset T \text{ et } \varepsilon > 0.$$

Le processus : $\eta(t) = \max\{\phi(t),\ \phi \in \Phi\}, \quad t \in T$ est un proccesus continu sur T et max-infiniment divisible, dont μ est la mesure exponentielle. Pour tout sous-ensemble fermé $S \subset T$, notons $\mathcal{F}_S$ la tribu engendrée par les variables aléatoires $\{\eta(s),\ s \in S\}$, et notons $\mathcal{P}_S$ la loi de la restriction η_S dans l'ensemble $\mathbb{C}(S)$ muni de $\mathcal{C}_S$ sa tribu borélienne. Pour $S_1, S_2 \subset T$ des sous-ensembles fermés disjoints, le coefficient de β-mélange entre $\mathcal{F}_{S_1}$ and $\mathcal{F}_{S_2}$ est donné par :

$$\beta(S_1, S_2) \quad = \quad \sup \Big\{ |\mathcal{P}_{S_1 \cup S_2}(C) - \mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2}(C)|;\ C \in \mathcal{C}_{S_1 \cup S_2} \Big\}.$$

Une autre définition (équivalente) du coefficient de β-mélange est :

$$\beta(S_1, S_2) \quad = \quad \frac{1}{2} \sup \Big\{ \sum_{i=1}^{I} \sum_{j=1}^{J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)| \Big\},$$

le supremum étant pris sur toutes les partitions $\{A_1, \ldots, A_I\}$ et $\{B_1, \ldots, B_J\}$ de Ω tel que pour $1 \le i \le I$, $A_i \in \mathcal{F}_{S_1}$ et pour $1 \le j \le J$, $B_j \in \mathcal{F}_{S_2}$. Le principal résultat du Chapitre 5 est le suivant.

Théorème 4.1. *Soit η un processus max-i.d. continu sur T de mesure exponentielle μ. Alors, pour tous sous-espaces fermés et disjoints $S_1, S_2 \subset T$ on a :*

$$\beta(S_1, S_2) \le 2 \int_{\mathcal{C}_0} \mathbb{P}[f \not<_{S_1} \eta,\ f \not<_{S_2} \eta]\, \mu(df).$$

Ce résultat peut être réécrit dans le cas où S_1 et S_2 sont dénombrables. Pour cela, pour tout $(s_1, s_2) \in T^2$, nous noterons μ_{s_1,s_2} la mesure exponentielle de $(\eta(s_1), \eta(s_2))$, on obtient le résultat suivant.

Corollaire 4.2. *Si S_1 et S_2 sont des sous-espaces fermés et disjoints, finis ou dénombrables, de T, alors :*

$$\beta(S_1, S_2) \le 2 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \int \mathbb{P}[\eta(s_1) \le y_1,\ \eta(s_2) \le y_2]\, \mu_{s_1,s_2}(dy_1 dy_2).$$

A partir de là, les résultats du Chapitre 5 ont été donnés sous la condition que η soit max-stable simple. Le qualificatif simple signifie que les marginales sont de loi Fréchet unitaire, c'est à dire :

$$\mathbb{P}[\eta(t) \le y] = \exp[-y^{-1}] 1_{\{y>0\}}, \quad y \in \mathbb{R},\ t \in T.$$

Un nouvel outil a été utilisé, le coefficient extrémal θ défini pour tout compact S par la relation suivante :

$$\mathbb{P}[\sup_{s \in S} \eta(s) \le y] = \exp[-\theta(S) y^{-1}], \quad y > 0.$$

Nous avons les propriétés suivantes.

Théorème 4.3. *Soit η un champ aléatoire max-stable simple continu sur T.*
Pour tout compact $S \subset T$, la quantité

$$C(S) = \mathbb{E}[\sup\{\eta(s)^{-1};\ s \in S\}]$$

est finie et de plus :

– *Pour tous compacts disjoints $S_1, S_2 \subset T$, on a :*

$$\beta(S_1, S_2) \leq 2\Big[C(S_1) + C(S_2)\Big]\Big[\theta(S_1) + \theta(S_2) - \theta(S_1 \cup S_2)\Big].$$

– *Pour $(S_{1,i})_{i\in I}$ et $(S_{2,j})_{j\in J}$ des familles dénombrables de compacts de T telles que $S_1 = \cup_{i\in I} S_{1,i}$ et $S_2 = \cup_{j\in J} S_{2,j}$ soient disjoints, on a :*

$$\beta(S_1, S_2) \leq 2\sum_{i\in I}\sum_{j\in J}\Big[C(S_{1,i}) + C(S_{2,j})\Big]\Big[\theta(S_{1,i}) + \theta(S_{2,j}) - \theta(S_{1,i} \cup S_{2,j})\Big].$$

En se mettant dans le cas où S_1 et S_2 sont dénombrables, et en adoptant la notation suivante :

$$\theta(s_1, s_2) = \theta(\{s_1, s_2\}), \quad s_1, s_2 \in T,$$

le théorème précédent nous donne le résultat suivant.

Corollaire 4.4. *Supposons que η soit un champ aléatoire max-stable simple continu sur T. Si S_1 et S_2 sont des sous-espaces fermés et disjoints, finis ou dénombrables, de T, alors :*

$$\beta(S_1, S_2) \leq 4\sum_{s_1\in S_1}\sum_{s_2\in S_2}[2 - \theta(s_1, s_2)].$$

Maintenant plaçons nous dans le cas où $T = \mathbb{Z}^d$ et notons $|h| = \max_{1\leq i\leq d}|h_i|$ la norme de $h \in \mathbb{Z}^d$. Bolthausen dans [10] a présenté une version du théorème de la limite centrale pour des processus stationnaires mélangeants. Aussi les résultats que nous avons obtenus et les critères de Bolthausen nous permettent d'aboutir au théorème suivant.

Théorème 4.5. *Supposons que η soit un champ aléatoire max-i.d. stationnaire sur $\mathbb{Z}^d$ de mesure exponentielle μ et considérons la quantité suivante :*

$$\gamma(h) = \int \mathbb{P}[\eta(0) \leq y_1,\ \eta(h) \leq y_2]\,\mu_{0,h}(dy_1 dy_2), \quad h \in \mathbb{Z}^d.$$

Soient $g : \mathbb{R}^p \to \mathbb{R}$ une fonction mesurable et $t_1, \ldots, t_p \in \mathbb{Z}^d$ tels que

$$\mathbb{E}[g(\eta(t_1), \ldots, \eta(t_p))^{2+\delta}] < \infty \;\; \textit{pour tout} \;\; \delta > 0.$$

Si les conditions suivantes sont vérifiées

$$\sum_{|h|\geq m}\gamma(h) = o(m^{-d}) \quad \textit{et} \quad \sum_{m=1}^{\infty} m^{d-1}\sup_{|h|\geq m}\gamma(h)^{\delta/(2+\delta)} < \infty,$$

alors le champ aléatoire stationnaire X défini par

$$X(t) = g(\eta(t_1 + t), \ldots, \eta(t_p + t)), \quad t \in \mathbb{Z}^d$$

vérifie le TCL.

Le Chapitre 5 se conclut par l'application de ce résultat afin de montrer la normalité asymptotique d'estimateurs du coefficient extremal.

5 Processus max-stables simples stationnaires et markoviens

Dans le Chapitre 6 nous avons déterminé tous les modèles de processus stochastiques sur les ensembles $T = \mathbb{Z}$ ou $T = \mathbb{R}$ max-stables simples stationnaires et vérifiant la propriété de Markov. Rappelons qu'un processus $X = (X(t))_{t\in T}$ est dit max-stable simple si et seulement si pour tout $n \geq 1$:

$$X \stackrel{\mathcal{L}}{=} n^{-1} \bigvee_{i=1}^{n} X_i$$

où $(X_i)_{i\geq 1}$ est une suite i.i.d de processus de même loi que X. Ici pour tout $t \in T$, $X(t)$ suit la loi de Fréchet unitaire, c'est à dire :

$$\mathbb{P}[X(t) \leq y] = \exp(-1/y), \quad \text{pour tout } y > 0.$$

Pour $(F_n)_{n\in\mathbb{Z}}$ une suite de variables aléatoires i.i.d. de loi Fréchet unitaire et $a \in [0, 1]$, on définira les processus max-stables maximum-autoregressifs d'ordre 1 (max-AR(1)) de paramètre a de la façon suivante :

$$X_a(t) = \begin{cases} (1-a) \bigvee_{n\leq t} a^{t-n} F_n & \text{si } a \in [0,1) \\ X_1(t) \equiv F_0 & \text{si } a = 1 \end{cases}, \quad t \in \mathbb{Z}.$$

On peut en effet remarquer que X_a vérifie la relation suivante qui lui confère aussi la propriété de Markov :

$$X_a(t+1) = \max(aX_a(t), (1-a)F_{t+1}), \quad t \in \mathbb{Z}.$$

Inversement $\check{X}_a = (X_a(-t))_{t\in\mathbb{Z}}$ le processus à temps renversé associé à X_a peut s'écrire :

$$\check{X}_a(t) = (1-a) \bigvee_{n\geq t} a^{n-t} \check{F}_n, \quad t \in \mathbb{Z},$$

où $(\check{F}_n)_{n\in\mathbb{Z}} = (F_{-n})_{n\in\mathbb{Z}}$, vérifie la relation de max-autoregressivité backward :

$$\check{X}_a(t-1) = \max(a\check{X}_a(t), (1-a)\check{F}_{t-1}), \quad t \in \mathbb{Z}.$$

D'une façon générale le processus à temps renversé d'un processus stationnaire et markovien est lui aussi stationnaire et markovien. Aussi notre principal résultat en ce qui concerne la caractérisation des processus max-stables simples stationnaires et markoviens est le théorème suivant.

Théorème 5.1. *Tout processus max-stable simple stationnaire vérifiant la propriété de Markov est égal en loi à un processus max-AR(1) ou à son processus à temps renversé associé.*

Pour prouver ce théorème nous avons eu besoin de la représentation des processus max-stables donnée par de Haan [30] : considérons $\eta = (\eta(t))_{t\in\mathbb{Z}}$ un processus max-stable simple, il existe un processus positif Y (non unique) tel que, pour tout $t \in \mathbb{Z}$, $\mathbb{E}[Y(t)] = 1$ et vérifiant la relation suivante :

$$\Big(\eta(t)\Big)_{t\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \Big(\bigvee_{i\geq 1} U_i Y_i(t)\Big)_{t\in\mathbb{Z}},$$

où $(Y_i)_{i\geq 1}$ est une suite de processus i.i.d. de même loi que Y, et $\{U_i, i \geq 1\}$ est un processus ponctuel de Poisson sur $(0, +\infty)$ de mesure d'intensité $u^{-2}du$ et indépendant de $(Y_i)_{i\geq 1}$. Aussi, les résultats du Chapitre 3 nous permettent d'écrire pour tout $t, t_1, \ldots, t_k \in \mathbb{Z}$ et $z, z_1, \ldots, z_k$ des réels strictement positifs, la formule suivante :

$$\begin{aligned}
& \mathbb{P}[\eta(t_1) \leq z_1, \ldots, \eta(t_k) \leq z_k \mid \eta(t) = z] \\
= \ & \mathbb{E}\big[1_{\{\vee_{i=1}^k \frac{Y(t_i)}{z_i} \leq \frac{Y(t)}{z}\}} Y(t)\big] \exp\Big(- \mathbb{E}\Big[\Big(\bigvee_{i=1}^{k} \frac{Y(t_i)}{z_i} - \frac{Y(t)}{z}\Big)^+\Big]\Big),
\end{aligned}$$

avec ici $(x)^+ = \max(x, 0)$. Cette formule nous a permis d'obtenir la proposition suivante :

Proposition 5.1. *Si η vérifie la propriété de Markov, alors pour tout $t_1 < t < t_2$, on a :*

$$\mathbb{P}[Y(t_1) = \alpha_{t,t_1} Y(t) \mid Y(t) > 0] = 1 \quad \textit{ou} \quad \mathbb{P}[Y(t_2) = \alpha_{t,t_2} Y(t) \mid Y(t) > 0] = 1,$$

où $\alpha_{t,t'}$ est l'infimum essentiel de $Y(t')/Y(t)$ conditionnellement à $Y(t) > 0$, i.e.

$$\alpha_{t,t'} = \inf\{c > 0;\ \mathbb{P}[Y(t')/Y(t) \leq c \mid Y(t) > 0] > 0\}.$$

Considérons les cônes suivants :

$$D_1 = \{f \in \mathcal{F};\ \forall t \in \mathbb{Z},\ f(t) = f(0)\},$$

$$D_0 = \{f \in \mathcal{F};\ \exists t_0 \in \mathbb{Z},\ \forall t \in \mathbb{Z},\ f(t) = f(t_0) 1_{\{t=t_0\}}\},$$

et pour $a \in (0, 1)$, le cône

$$D_a = \{f \in \mathcal{F};\ \exists t_0 \in \mathbb{Z},\ \forall t \in \mathbb{Z},\ f(t) = f(t_0) a^{t-t_0} 1_{\{t \geq t_0\}}\}.$$

Nous avons la propriété suivante :

Proposition 5.2. *Soit η un processus max-stable simple. Si η est stationnaire, alors pour tout $a \in [0, 1]$ les assertions suivantes sont équivalentes :*

i) η a la même loi que le processus Max-AR(1) X_a défini précédemment ;

ii) $\mathbb{P}[Y \in D_a] = 1$.

Ces deux propositions nous permettent de démontrer le Théorème 5.1. Le Chapitre 6 a été conclu par une géneralisation du résultat obtenu dans le cas continu. Pour $a \in (0, 1)$, notons par g_a la fonction $t \mapsto -\log(a) a^t 1_{\{t \geq 0\}}$. On peut voir que $\int_{\mathbb{R}} g_a(t) dt = 1$. Considérons alors le processus suivant :

$$Z_a(t) = \bigvee_{i\geq 1} U_i g_a(t - T_i), \quad t \in \mathbb{R},$$

où $\{(U_i, T_i);\ i \geq 1\}$ est un processus ponctuel de Poisson sur $(0, +\infty) \times \mathbb{R}$ d'intensité $u^{-2} du dt$. On définira de façon similaire $\check{Z}_a$ le processus à temps renversé :

$$\check{Z}_a(t) = \bigvee_{i\geq 1} U_i \check{g}_a(t - T_i), \quad t \in \mathbb{R},$$

avec $\check{g}_a(t) = -\log(a) a^{-t} 1_{\{t<0\}} = g_a(-t^-)$, $\check{g}_a$ est càd-làg. Pour $a = 1$, $Z_1 = \check{Z}_1$ est un processus à trajectoire constante tel que $Z_1(0)$ suive une loi de Fréchet unitaire. Nous avons alors la propriété suivante :

Théorème 5.2. *Tout processus max-stable simple stationnaire* $\eta = (\eta(t))_{t\in\mathbb{R}}$ *à trajectoires càd-làd vérifiant la propriété de Markov est égal en loi au processus* Z_a *ou au processus* $\check{Z}_a$ *pour un certain* $a \in (0, 1]$.

Chapitre 1

Préliminaire

1.1 Processus Ponctuel de Poisson

Soient E un espace localement compact à base dénombrable et $\mathcal{E}$ sa tribu borélienne.

Définition 1.1.1. *Soit $(x_n)_{n\geq 1}$ une suite de points de E. On appelle mesure ponctuelle sur E, une mesure m de la forme $m = \sum_{n=1}^{\infty} \delta_{x_n}$ où δ_{x_n} est la mesure de Dirac en x_n.*

Nous allons désigner par $M_p(E)$ l'espace de toutes les mesures ponctuelles définies sur E et le munir de $\mathcal{M}_p(E)$ la plus petite tribu qui pour tout borélien $F \in \mathcal{E}$ rend mesurable l'application $m \in (M_p(E), \mathcal{M}_p(E)) \longmapsto m(F) \in ([0,\infty], \mathcal{B}_{[0,\infty]})$. On définit les processus ponctuels aléatoires de la façon suivante.

Définition 1.1.2. *On appelle processus ponctuel aléatoire toute application mesurable $N : (\Omega, \mathcal{F}, \mathbb{P}) \to (M_p(E), \mathcal{M}_p(E))$.*

A partir de là, on définit alors les processus ponctuels de Poisson de la façon suivante.

Définition 1.1.3. *Soient μ une mesure de Radon (finie sur tout compact) sur $\mathcal{E}$, et N un processus ponctuel. On dit que N est un processus ponctuel de Poisson de mesure d'intensité μ si :*

i) Pour tout $F \in \mathcal{E}$ et pour tout entier positif k :

$$\mathbb{P}[N(F) = k] = \begin{cases} \exp\{-\mu(F)\}(\mu(F))^k/k! & \textit{si } \mu(F) < \infty \\ 0 & \textit{si } \mu(F) = \infty \end{cases}.$$

ii) Pour $(F_k)_{k\geq 1}$ une famille d'ensembles disjoints de $\mathcal{E}$, $(N(F_k))_{k\geq 1}$ forme une famille de variables aléatoires indépendantes.

Le théorème suivant, sera très souvent utilisé dans les démonstrations faisant intervenir les processus ponctuels de Poisson.

Théorème 1.1.4. *Soit Π un processus ponctuel de Poisson de mesure d'intensité μ, σ-fini sur l'espace d'état E, et soit $f : E \to T$ une fonction mesurable telle que la mesure image de μ par f soit une mesure de Radon. Alors $f(\Pi)$ est un processus de Poisson sur T de mesure d'intensité $\mu \circ f^{-1}$.*

1.2 Quelques notions sur les lois des valeurs extêmes univariés

Dans cette section, nous rappelons quelques notions de base de théorie des valeurs extrêmes univariées.

Définition 1.2.1. *Soient X une v.a. et $(X_i)_{i\in\mathbb{N}}$ une suite i.i.d de v.a. de même loi que X. On dit que X est max-stable si, pour tout $n \in \mathbb{N}$, il existe des réels $a_n > 0$ et b_n tels que :*

$$\max_{i\leq n}(a_n^{-1}(X_i - b_n)) \stackrel{\mathcal{L}}{=} X.$$

On peut remarquer que cette définition est aussi valable pour les processus stochastiques.

Proposition 1.2.1. *Si X est une v.a. max-stable, alors pour tout réels μ et $\sigma > 0$, $\sigma X + \mu$ est aussi max-stable.*

Preuve :
Une fonction F est la fonction de répartition d'une v.a. max-stable ssi $\forall n \in \mathbb{N}$, il existe des réels $a_n > 0$ et b_n tels que $F^n(a_n x + b_n) = F(x)$. En effet, soit $(X_i)_{i\in\mathbb{N}}$ une suite i.i.d de v.a. de même loi que X, pour tout réel $a > 0$ et b on a :

$$\begin{aligned}\mathbb{P}[\max_{1\leq i\leq n}(a^{-1}(X_i - b)) \leq x] &= \prod_{i=1}^{n}\mathbb{P}[(a^{-1}(X_i - b)) \leq x] \\ &= (\mathbb{P}[(a^{-1}(X - b_n)) \leq x])^n \\ &= F^n(ax + b).\end{aligned}$$

Or, si X est une v.a. max-stable, il existe alors des réels $a_n > 0$ et b_n tels que $\max_{i\leq n}(a_n^{-1}(X_i - b_n))$ et X ont même loi. Finalement, on a :

$$\begin{aligned}\mathbb{P}[\sigma X + \mu \leq x] &= \mathbb{P}[X \leq \frac{(x-\mu)}{\sigma}] \\ &= (\mathbb{P}[X \leq a_n\frac{(x-\mu)}{\sigma} + b_n])^n \\ &= (\mathbb{P}[\sigma X + \mu \leq a_n x + (\sigma b_n - a_n\mu + \mu)])^n.\end{aligned}$$

Ainsi $\sigma X + \mu$ et $\max_{i\leq n}\{a_n^{-1}((\sigma X_i + \mu) - (\sigma b_n - a_n\mu + \mu))\}$ ont même loi, donc $\sigma X + \mu$ est max-stable. □

Théorème 1.2.2. *Les trois lois suivantes sont max-stables :*

i) La loi de Gumbel qui a pour fonction de répartition :

$$F(x) = \exp(-e^{-x}).$$

ii) La loi de Fréchet de paramètre $\theta > 0$, qui a pour fonction de répartition :

$$F(x;\theta) = \exp((-x)^{\theta})1_{]0;+\infty)}(x).$$

iii) La loi de Weibull à paramètre $\theta > 0$, qui a pour fonction de répartition :

$$F(x;\theta) = 1 - e^{-(x^\theta)}1_{]0;+\infty)}(x).$$

Preuve :
• Dans le cas de la loi de Gumbel, on a :

$$\begin{aligned} F^n(x - \ln n) \quad &= \exp(-ne^{-x+\ln n}) \\ &= \exp(-e^{-x}) \\ &= F(x) \end{aligned}$$

Ici $a_n = 0$, et $b_n = \ln n$, la loi de Gumbel est donc bien max-stable.

• Dans le cas de la loi de Fréchet, on a :

$$\begin{aligned} F^n(n^{\frac{1}{\theta}}x;\theta) \quad &= \exp(n(-n^{\frac{1}{\theta}}x)^{-\theta})1_{]0;+\infty)}(n^{\frac{1}{\theta}}x) \\ &= \exp((-x)^{-\theta})1_{]0;+\infty)}(x) \\ &= F(x;\theta). \end{aligned}$$

Ici $a_n = n^{\frac{1}{\theta}}$, et $b_n = 0$, la loi de Fréchet est donc bien max-stable.

• Dans le cas de la loi de Weibull, nous allons plutôt travailler avec sa queue, on a :

$$\begin{aligned} Q^n(n^{-\frac{1}{\theta}}x;\theta) \quad &= (1 - F(n^{-\frac{1}{\theta}}x;\theta))^n \\ &= 1_{]0;+\infty)}(n^{-\frac{1}{\theta}}x)\exp(-n(n^{-\frac{1}{\theta}}x)^\theta) \\ &= e^{-(x^\theta)}1_{]0;+\infty)}(x) \\ &= Q(x;\theta)\ . \end{aligned}$$

Ici $a_n = n^{-\frac{1}{\theta}}$, et $b_n = 0$, la loi de Weibull est donc bien max-stable. □
Comme le suggère la Proposition 1.2.1 toute transformation affine de ces loi demeure max-stable. Notons au passage que si la variable aléatoire X suit une loi de Gumbel, alors la variable aléatoire e^X suit quant à elle une loi de Fréchet.

Théorème 1.2.3 (Fisher & Tippet 1928)**.** *Toutes les lois max-stables sont à une transformation affine près du type Gumbel, Weibull et Fréchet.*

1.3 Représentation de processus max-stables

Par représentation de processus max-stables il est en fait question de constructions de processus max-stables à partir de processus ponctuels de Poisson. Dans un cadre bien plus général Giné et al. [44] nous ont donné une relation entre processus ponctuels de Poisson et processus max-stables. Mais ici nous allons plutôt rappeler la méthode de construction de Smith [70]. Pour S un ensemble mesurable, soit $\Phi = \sum_{i\geqslant 1} \delta_{\{\xi,s_i\}}$ un processus ponctuel de Poisson sur $(0,\infty)\times S$

de mesure d'intensité $\xi^{-2}d\xi\nu(ds)$, où ν est une mesure positive sur S. Pour T un ensemble paramétrique quelconque, soit $f : S \times T \to [0, \infty)$ une fonction positive mesurable telle que :

$$\int_S f(s,t)\nu(ds) = 1, \quad \forall t \in T.$$

Alors le processus η défini par

$$\eta(t) = \bigvee_{i\in\mathbb{N}} \xi_i f(s_i, t), \quad t \in T.$$

est un processus max-stable sur T. Le processus est max-stable simple et plus généralement pour I un sous ensemble de T, on obtient :

$$P(\eta(t) \leq y_t, t \in I) = \exp\left[-\int_S \max_{t\in I}\left\{\frac{f(s)}{y_t}\nu(ds)\right\}\right].$$

Exemple 1 : Pour $S = \mathbb{R}^{\mathbb{R}^d}$ et $T = \mathbb{R}^d$, si l'on choisit le cas où ν est la loi d'un processus gaussien centré à accroissement stationnaire X et $f : (g,t) \longmapsto e^{g(t)-Var(X(t))/2}$, dans ce cas, nous avons bien :

$$\int_S f(s,t)\nu(ds) = \mathbb{E}[e^{X(t)-Var(X(t))/2}] = 1.$$

De plus pour $(X_i)_{i\in\mathbb{N}}$ une suite de processus i.i.d de même loi que X, la représentation

$$\eta(t) = \bigvee_{i\in\mathbb{N}} \xi_i f(s_i, t) \overset{\mathcal{L}}{=} \bigvee_{i\in\mathbb{N}} \xi_i e^{X_i(t)-Var(X(t))/2}, \quad t \in T,$$

correspond alors à une représentation bien étudiée par Kabluchko et al. [51].

Exemple 2 : Pour $S = \mathbb{R}^{\mathbb{R}^d} \times \mathbb{R}^d$, soient F un processus stochastique de loi $\mathbb{P}_F$ tel que $\mathbb{E}\left[\int_{\mathbb{R}^d} F(t)dt\right] = 1$, $\nu = \mathbb{P}_F ds$ et f la fonction définie par, $f(g,s,t) = g(t-s)$, $t \in \mathbb{R}^d$. On a

$$\int\int_{\mathbb{R}^d} f(g,s,t)ds\mathbb{P}_F(dg) = \mathbb{E}\left[\int_{\mathbb{R}^d} F(t-s)ds\right] = 1.$$

Aussi pour $(F_i)_{i\in\mathbb{N}}$ une suite de processus i.i.d de même loi que F et $\sum_{i\in\mathbb{N}} \delta_{\{s_i\}}$ un processus ponctuel de Poisson d'intensité la mesure de Lebesgue sur $\mathbb{R}^d$, la représentation suivante

$$\eta(t) = \bigvee_{i\in\mathbb{N}} \xi_i f(s_i, t) \overset{\mathcal{L}}{=} \bigvee_{i\in\mathbb{N}} \xi_i F_i(t - s_i), \quad t \in T,$$

est appelée “mixed moving maxima process“ et est aussi souvent rencontrée dans la littérature [51, 56].

Chapitre 2

Extremes of independent stochastic processes : a point process approach

Abstract

For each $n \geq 1$, let $\{X_{in}, \quad i \geqslant 1\}$ be independent copies of a nonnegative continuous stochastic process $X_n = (X_n(t))_{t \in T}$ indexed by a compact metric space T. We are interested in the process of partial maxima

$$\tilde{M}_n(u, t) = \max\{X_{in}(t), 1 \leqslant i \leqslant [nu]\}, \quad u \geq 0, \ t \in T.$$

where the brackets $[\cdot]$ denote the integer part. Under a regular variation condition on the sequence of processes X_n, we prove that the partial maxima process $\tilde{M}_n$ weakly converges to a superextremal process $\tilde{M}$ as $n \to \infty$. We use a point process approach based on the convergence of empirical measures. Properties of the limit process are investigated : we characterize its finite-dimensional distributions, prove that it satisfies an homogeneous Markov property, and show in some cases that it is max-stable and self-similar. Convergence of further order statistics is also considered. We illustrate our results on the class of log-normal processes in connection with some recent results on the extremes of Gaussian processes established by Kabluchko.

2.1 Introduction

A classical problem in extreme value theory (EVT) is to determine the asymptotic behavior of the maximum of independent and identically distributed (i.i.d.) random variables $(Z_i)_{i \geq 1}$. What are the assumptions that ensure the weak convergence of the rescaled maximum

$$\max_{1 \leq i \leq n} \frac{Z_i - b_n}{a_n}, \quad a_n > 0, \ b_n \in \mathbb{R},$$

and what are the possible limit distributions ? These questions were at the basis of the development of EVT and found its answer in the Theorem by Fisher and Tippett [40] characterizing all the max-stable distributions and in the description of their domain of attraction by Gnedenko [45] and de Haan [28]. Another stimulating point of view developped by Lamperti [52] is to introduce a time variable and consider the asymptotic behavior of the partial maxima

$$\max_{1\leq i\leq [nu]} \frac{Z_i - b_n}{a_n}, \quad u \geq 0.$$

The corresponding limit process is known as an extremal process : it is a pure jump Markov process wich is also max-stable, see Resnick [61].

Since then, EVT has known many developments. Among several other directions, the extension of the theory to multivariate and spatial settings is particularly important, as well as the statistical issues raised by the applications on real data sets. For excellent reviews of such developments, the reader is invited to refer to the monographies by Resnick [62], de Haan and Fereira [31] or Beirlant *et al* [6] and the references therein.

Our purpose here is to focus on the functional framework and investigate the asymptotic behavior of the partial maxima processes based on a doubly infinite array of independent random processes. Let T be a compact metric space and, for each $n \geq 1$, let $\{X_{in},\ i \geqslant 1\}$ be independent copies of a sample continuous stochastic process $(X_n(t))_{t\in T}$. Without loss of generality, we will always suppose that X_n is nonnegative (otherwise consider $X_n'(t) = e^{X_n(t)}$) and we denote by $\mathbb{C}^+ = \mathbb{C}^+(T)$ be the set of nonnegative continuous functions on T. We are mainly interested in the process of pointwise maxima

$$M_n(t) = \max\{X_{in}(t),\ 1 \leqslant i \leqslant n\}, \quad t \in T, \tag{2.1}$$

and the process of partial maxima

$$\tilde{M}_n(u,t) = \max\{X_{in}(t),\ 1 \leqslant i \leqslant [nu]\}, \quad u \geq 0,\ t \in T. \tag{2.2}$$

We use the convention that the maximum over an empty set is equal to 0, so $\tilde{M}_n(0,t) \equiv 0$. Clearly, we also have $\tilde{M}_n(1,t) = M_n(t)$. In this framework, the parameter $t \in T$ is thought as a space parameter and $u \in [0,+\infty)$ as a time variable.

Our approach relies on the convergence of the following empirical measures

$$\beta_n = \sum_{i=1}^{n} \delta_{X_{in}} \quad \text{and} \quad \tilde{\beta}_n = \sum_{i\geq 1} \delta_{(X_{in}, i/n)} \tag{2.3}$$

on $\mathbb{C}^+$ and $\mathbb{C}^+ \times [0,+\infty)$ respectively. The maxima processes M_n and $\tilde{M}_n$ can be written as functionals of the empirical measure β_n and $\tilde{\beta}_n$ respectively. We will prove the continuity of the underlying functionals and then use the continuous mapping Theorem to deduce convergence of the maxima processes from convergence of empirical measures.

Connections between EVT and point processes are well known. In the case when the underlying state space is locally compact, the following result holds (see Proposition 3.21 in[62]) : if $n\mathbb{P}[X_n \in \cdot\,]$ vaguely converges to some measure μ, the empirical measures β_n and $\tilde{\beta}_n$ converge to Poisson random measures with intensity μ and $\mu \otimes \ell$ respectively, ℓ being the Lebesgue measure on $[0,+\infty)$. However, in our framework the state space $\mathbb{C}^+$ is not locally compact and we need a suitable generalization of the above result. To this aim, we follow the approach by Davis and Mikosch [22] based on the notion of boundedly finite measures and $\sharp$-weak convergence detailed by Daley and Vere-Jones in [20].

The paper is organized as follows. In section 2, we introduce the technical material needed on boundedly finite measures, $\sharp$-weak convergence and convergence of empirical measures. Our main result is the convergence of the partial maxima process $\widetilde{M}_n$ which is stated and proved in section 3. Then, in section 4, we investigate some properties of the limit process M, known as a superextremal process. A brief extension of our results to further order statisctics is considered in section 5. The last section is devoted to an application of our results to the class of log-normal processes, based on a recent work by Kabluchko [49] on the extremes of Gaussian processes.

2.2 Preliminaries on boundedly finite measures and point processes

In this section, we present general results on boundedly finite measures and point processes that will be useful in the sequel. The reader should refer to Appendix 2.6 in Daley and Vere-Jones [20] or to section 2 of Davis and Mikosch [22]. This has also closed connections with the theory of regular variations, see Hult and Lindskog [46, 47].

2.2.1 Boundedly finite measures

Let (E, d) be a complete separable metric space and $\mathcal{E}$ the Borel σ-algebra of E. We denote by $M_b(E)$ the set of all finite measures on $(E, \mathcal{E})$. A sequence of finite measures $\{\mu_n, n \geqslant 1\}$ is said to converge weakly to $\mu \in M_b(E)$ if and only if $\int f d\mu_n \to \int f d\mu$ for all bounded continuous function f on E. We write $\mu_n \stackrel{w}{\to} \mu$ to denote weak convergence. It is well known that this notion of convergence is metrized by the Prokhorov metric

$$p(\mu_1, \mu_2) = \inf\{\varepsilon > 0;\ \forall A \in \mathcal{E},\ \mu_1(A) \leqslant \mu_2(A^\varepsilon) + \varepsilon \ \text{ and } \ \mu_2(A) \leqslant \mu_1(A^\varepsilon) + \varepsilon\}$$

where $A^\varepsilon = \{x \in E;\ \exists a \in A,\ d(a, x) < \varepsilon\}$ is the ε-neighborhood of A. Endowed with this metric, $M_b(E)$ is a complete separable metric space.

A measure μ on $(E, \mathcal{E})$ is called boundedly finite if it assigned finite measure to bounded sets, *i.e.* $\mu(B) < \infty$ for all bounded $B \in \mathcal{E}$. Let $M_b^\sharp(E)$ be the set of all boundedly finite measures μ on $(E, \mathcal{E})$. A sequence of boundedly finite measures $\{\mu_n, n \geqslant 1\}$ is said to converge $\sharp$-weakly to $\mu \in M_b^\sharp(E)$ if and only if $\int f d\mu_n \to \int f d\mu$ for all bounded continuous function f on E with bounded support. There exists a metric $p^\sharp$ on $M_b^\sharp(E)$ that is compatible with this notion of $\sharp$-weak convergence and that makes $M_b^\sharp(E)$ a complete and separable metric space. Such a metric can be constructed as follows. Fix an origin $e_0 \in E$ and, for $r > 0$, let $\bar{B}_r = \{x \in E;\ d(x, e_0) \leq r\}$ be the closed ball of center e_0 and radius r. For any $\mu_1, \mu_2 \in M_b^\sharp(E)$, let $\mu_1^{(r)}$ and $\mu_2^{(r)}$ be the restriction of μ_1 and μ_2 to $\bar{B}_r$. Note that μ_1 and μ_2 are finite measures on $\bar{B}_r$ and denote by p_r

the Prokhorov metric on $M_b(\bar{B}_r)$. Define

$$p^\sharp(\mu_1,\mu_2) = \int_0^\infty e^{-r}\frac{p_r(\mu_1^{(r)},\mu_2^{(r)})}{1+p_r(\mu_1^{(r)},\mu_2^{(r)})}dr.$$

There are several equivalent characterizations of $\sharp$-weak convergence. Let $\{\mu_n, n\geqslant 1\}$ and μ be boundedly finite measures on E. The following statements are equivalent :

i) $\int f d\mu_n \to \int f d\mu$ for all f bounded continuous real valued function on E with bounded support ;

ii) $p^\sharp(\mu_n,\mu)\to 0$;

iii) there exists a sequence $r_k \nearrow +\infty$ such that, for all $k\geqslant 1$, $\mu_n^{(r_k)} \overset{w}{\to} \mu^{(r_k)}$;

iv) $\mu_n(B)\to\mu(B)$ for all bounded $B\in\mathcal{E}$ such that $\mu(\partial B)=0$.

We write $\mu_n \overset{w^\sharp}{\to} \mu$ to denote $\sharp$-weak convergence.

A boundedly finite point measure on E is a measure μ of the form

$$\mu = \sum_{i\in I}\delta_{x_i}$$

with $\{x_i, i\in I\}$ a finite or countable family of points in E such that any bounded set $B\subseteq E$ contains at most a finite number of the x_i's. The set of boundedly finite point measures on E is denoted by $M_{(b,p)}^\sharp(E)$. It is a closed subset of $M_b^\sharp(E)$ and hence is complete and separable when endowed with the induced metric.

The following characterization of $\sharp$-weak convergence of boundedly finite point measures will be useful. Let $\{\mu_n = \sum_i \delta_{x_i^n}, n\geqslant 1\}$ and $\mu = \sum_i\delta_{x_i}$ be elements of $M_{(b,p)}^\sharp(E)$. Then $\mu_n \overset{w^\sharp}{\to}\mu$ if and only if there exists some sequence $r_k\nearrow+\infty$ such that for all $k\geq 1$, there exists $N\geqslant 0$ and $m\geqslant 0$ such that

$$\forall n\geqslant N,\quad \mu_n^{(r_k)} = \sum_{i=1}^m \delta_{x_i^{n,k}} \text{ and } \mu^{(r_k)} = \sum_{i=1}^m\delta_{x_i^k}$$

for some points $\{x_i^{n,k}, x_i^k;\ 1\leq i\leq m,\ n\geq N\}$ in $\bar{B}_{r_k}$ such that $x_i^{n,k}\to x_i^k$.

2.2.2 Convergence of the empirical measures

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. A boundedly finite point process on E is a measurable mapping $N:\Omega\to M_{(b,p)}^\sharp(E)$. A typical example of boundedly finite point process on E is a Poisson point process Π_ν with intensity $\nu\in M_b^\sharp(E)$. We will also consider the following empirical point processes β_n and $\tilde\beta_n$ defined by Equation (2.3), where, for each $n\geq 1$, $\{X_{in},\ i\geq 1\}$ are independent copies of an E-valued random variable X_n. The random variable β_n is a finite point process on E, while $\tilde\beta_n$ is a boundedly finite point process on $\tilde E = E\times[0,+\infty)$ endowed with the metric

$$\tilde d((x_1,u_1),(x_2,u_2)) = d(x_1,x_2)+|u_2-u_1|,\quad x_1,x_2\in E,\ u_1,u_2\in[0,+\infty).$$

The following Proposition will play a key role in the sequel.

Proposition 2.2.1. *The following statements are equivalent, where the limits are taken as $n \to \infty$*

i) $n\mathbb{P}[X_n \in \cdot\,] \overset{w^\sharp}{\to} \nu$ *;*

ii) $\beta_n \Rightarrow \Pi_\nu$ *with* Π_ν *a Poisson point process on* E *with intensity* ν *and* $\Rightarrow$ *standing for weak convergence in* $M^\sharp_{(b,p)}(E)$.

iii) $\tilde{\beta}_n \Rightarrow \tilde{\Pi}_\nu$ *with* $\tilde{\Pi}_\nu$ *Poisson point process on* $E \times [0,+\infty)$ *with intensity* $\mu \otimes \ell$, ℓ *being the Lebesgue measure on* $[0,+\infty)$ *and* $\Rightarrow$ *standing for weak convergence in* $M^\sharp_{(b,p)}(E \times [0,+\infty))$.

The proof of this Proposition follows by an adapation of Proposition 3.21 in Resnick [62] which states a similar result in the case of a locally compact state space and in terms of vague convergence. As noticed by Davis and Mikosch [22], the proof remains valid for a complete separable metric space E if we change vague convergence by $\sharp$-weak convergence. See also Theorem 4.3 in Davydov, Molchanov and Zuyev [27].

In the terminology of Hult and Lindskog [46, 47], the condition i) in Proposition 2.2.1 means that the sequence X_n is *regularly varying*. A particularly important case is when E is endowed with a strucure of cone, *i.e.* a multiplication by positive scalars. If there exists a sequence $(a_n)_{n\geq 1}$ of positive reals such that

$$n\mathbb{P}[a_n^{-1}X_1 \in \cdot\,] \overset{w^\sharp}{\to} \nu$$

for some nonzero $\nu \in M_b^\sharp(E)$, then the random variable X_1 is said to be *regularly varying*. Under some technical assumptions on the structure of the convex cone E (*e.g.* continuity properties of the multiplication), the limit measure ν is proved to be homogeneous of order $-\alpha < 0$, *i.e.*

$$\nu(\lambda A) = \lambda^{-\alpha}\nu(A), \quad \lambda > 0, A \in \mathcal{E}.$$

Furthermore, the sequence a_n is regularly varying with index $1/\alpha$. In this framework, if $\{X_i,\ i \geq 1\}$ is an i.i.d. sequence of regularly varying random variables and $X_{in} = a_n^{-1}X_i$, then condition i) is equivalent to

$$n\mathbb{P}[a_n^{-1}X_1 \in \cdot\,] \overset{w^\sharp}{\to} \nu$$

where the limit measure ν is homogeneous of order $-\alpha < 0$ and

$$M_n(u,t) = a_n^{-1} \max_{1\leq i\leq [nu]} X_i(t), \quad u \geq 0,\ t \in T.$$

2.3 Extremes of independent stochastic processes

We go back to our original problem where the X_{in}'s are non negative continuous processes on T. We endow the set $\mathbb{C}^+ = \mathbb{C}^+(T)$ of non negative continuous functions on T with the uniform norm $\|x\| = \sup_{t\in T}|x(t)|$, $x \in \mathbb{C}^+$. It turns out that, when working with maxima, the uniform metric is not adaptated, mainly because sets such as $\{x \in \mathbb{C}^+; \|x\| \geq \varepsilon\}$ are not bounded for this metric. For this reason, we introduce $\overline{\mathbb{C}}_0^+ = (0,+\infty] \times \mathbb{S}_{\mathbb{C}^+}$ where $\mathbb{S}_{\mathbb{C}^+} = \{x \in \mathbb{C}^+ : \|x\|_\infty = 1\}$ is the unit sphere. We define the metric

$$d((r_1,s_1),(r_2,s_2)) = |1/r_1 - 1/r_2| + \|s_1 - s_2\|, \quad (r_1,s_1),(r_2,s_2) \in \overline{\mathbb{C}}_0^+.$$

The metric space $(\overline{\mathbb{C}}_0^+, d)$ is complete and separable. Define also $\mathbb{C}_0^+ = \mathbb{C}^+ \setminus \{0\}$ and consider the "polar decomposition" :

$$T : \begin{cases} \mathbb{C}_0^+ & \to \quad (0,\infty) \times \mathbb{S}_{\mathbb{C}^+} \\ x & \mapsto \quad (\|x\|, x/\|x\|) \end{cases}$$

The mapping T is an homeomorphism and we identify in the sequel $\mathbb{C}_0^+$ and $(0,+\infty) \times \mathbb{S}_{\mathbb{C}^+}$. In this metric, a subset B of $\mathbb{C}_0^+$ is bounded if and only if it is bounded away from zero, *i.e.* included in a set of the form $\{x \in \mathbb{C}^+ ; \|x\| \geq \varepsilon\}$ for some $\varepsilon > 0$.

2.3.1 Spatial maximum process

For the sake of clarity, we present first our results on the spatial maximum process M_n defined by Equation (2.1).

Theorem 2.3.1. *Assume that $n\mathbb{P}[X_n \in \cdot\,] \xrightarrow{w^\sharp} \nu$ in $M_b^\sharp(\overline{\mathbb{C}}_0^+)$ and that $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$. Then, the process M_n weakly converges in $\mathbb{C}^+$ as $n \to \infty$ to the process M defined by*

$$M(t) = \sup_{i \geq 1} Y_i(t), \quad t \in T, \tag{2.4}$$

where $\sum_{i\geq 1} \delta_{Y_i}$ is a Poisson point process on $\mathbb{C}_0^+$ with intensity ν.

The proof of Theorem 2.3.1 relies on the following Lemma. With a slight abuse of notation, we define

$$M_{(b,p)}^\sharp(\mathbb{C}_0^+) = \{\mu \in M_{(b,p)}^\sharp(\overline{\mathbb{C}}_0^+) : \mu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0\}.$$

Note $M_{(b,p)}^\sharp(\mathbb{C}_0^+)$ is an open subset of $M_{(b,p)}^\sharp(\overline{\mathbb{C}}_0^+)$.

Lemma 2.3.2. *The mapping $\theta : M_{(b,p)}^\sharp(\mathbb{C}_0^+) \to \mathbb{C}^+$ defined by $\theta(0) \equiv 0$ and*

$$\theta\Big(\sum_{i \in I} \delta_{(r_i, s_i)}\Big) = \Big(t \mapsto \sup\{r_i s_i(t);\ i \in I\}\Big)$$

is well-defined and continuous.

Proof of Lemma 2.3.2:

- First, we show that the mapping θ is well defined. This is not obvious since the pointwise supremum of a countable family of functions is not necessarily finite nor continuous.
  For $\varepsilon > 0$, let $\mu^{(\varepsilon)}$ be the restriction of μ on the set $\bar{B}^\varepsilon = [\varepsilon, \infty] \times \mathbb{S}_{\mathbb{C}}$. Since $\bar{B}^\varepsilon$ is bounded for the metric d, the point measure $\mu^{(\varepsilon)}$ has only a finite number of atoms and therefore $\theta(\mu^{(\varepsilon)})$ is the maximum of a finite number of non negative continuous functions and is hence a non negative continuous function.
  Furthermore, for all $t \in T$, $|\theta(\mu^{(\varepsilon)})(t) - \theta(\mu)(t)| \leqslant \varepsilon$. So $\theta(\mu)$ is the uniform limit as $\varepsilon \to 0$ of the continuous functions $\theta_\varepsilon(\mu)$, and hence $\theta(\mu) \in \mathbb{C}^+$ and satisfies

$$\|\theta(\mu^{(\varepsilon)}) - \theta(\mu)\| \leqslant \varepsilon. \tag{2.5}$$

– Second, we show that the mapping θ is continuous.
Let $\{\mu_n = \sum_{i\in I_n} \delta_{x_i^n}, \quad n \geqslant 1\}$ be a sequence of measures in $M^\sharp_{(b,p)}(\mathbb{C}_0^+)$ converging to $\mu = \sum_{i\in I} \delta_{x_i}$. For all $\varepsilon > 0$, there exists $\varepsilon' < \varepsilon$, $N \geqslant 0$ and $m \geqslant 0$ such that

$$\forall n \geqslant N, \quad \mu_n^{(\varepsilon')} = \sum_{i=1}^m \delta_{x_i^{n\varepsilon'}} \text{ and } \mu^{(\varepsilon')} = \sum_{i=1}^m \delta_{x_i^{\varepsilon'}}$$

with $x_i^{n\varepsilon'}, x_i^{\varepsilon'} \in \bar{B}^{\varepsilon'}$ and $x_i^{n\varepsilon'} \to x_i^{\varepsilon'}$ as $n \to \infty$. Clearly, this implies that $\theta(\mu_n^{(\varepsilon')})$ converges in $\mathbb{C}^+$ to $\theta(\mu^{(\varepsilon')})$ as $n \to \infty$. Hence there exists an integer n_0 such that $\|\theta(\mu_n^{(\varepsilon')}) - \theta(\mu^{(\varepsilon')})\| \leq \varepsilon$ for all $n \geq n_0$.
Then, thanks to Equation (2.5), we obtain for $n \geq n_0$

$$\|\theta(\mu_n) - \theta(\mu)\| \leq \|\theta(\mu_n) - \theta(\mu_n^{(\varepsilon')})\| + \|\theta(\mu_n^{(\varepsilon')}) - \theta(\mu^{(\varepsilon')})\| + \|\theta(\mu^{(\varepsilon')}) - \theta(\mu))\| \leq 3\varepsilon.$$

This proves that $\theta(\mu_n) \to \theta(\mu)$ as $n \to \infty$ and that the mapping θ is continuous.

□

Proof of Theorem 2.3.1:
Under the assumption $n\mathbb{P}[X_n \in \cdot\,] \stackrel{w^\sharp}{\to} \nu$, we know from Proposition 2.2.1 that the empirical measure β_n defined by Equation (2.3) weakly converges in $M_{(b,p)}(\overline{\mathbb{C}}_0^+)$ to a Poisson point process Π_ν with intensity ν. The assumption $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$ ensures that Π_ν lies almost surely in $M_{(b,p)}(\mathbb{C}_0^+)$. Note that $M_n = \theta(\beta_n)$, and according to Lemma 2.3.2, the mapping θ is continuous. So the continuous mapping Theorem (see *e.g.* Theorem 5.1 in Billingsley [8]) entails $\theta(\beta_n) \Rightarrow \theta(\Pi_\nu)$ which is equivalent to $M_n \Rightarrow M$. □

2.3.2 Spatio-temporal maximum process

We consider now convergence of the space-time process $\tilde{M}_n$ defined by Equation (2.2).

For fixed $u \geq 0$, the space process $t \mapsto \tilde{M}_n(u,t)$ is sample continuous and non negative, i.e. a random elements of $\mathbb{C}^+$. Furthermore, the time process $u \mapsto \tilde{M}_n(u,\cdot)$ can be seen as a $\mathbb{C}^+$-valued càd-làg process on $[0,+\infty)$; it is indeed constant on intervals of the form $[k/n, k/n+1/n)$, $k \in \mathbb{N}$. Hence, we will consider the process $\tilde{M}_n$ as a random element of the Skohorod space $\mathbb{D}([0,+\infty),\mathbb{C}^+)$ endowed with the J_1-topology (see for example Ethier and Kurtz [39] for the definition and properties of Skohorod space).

Theorem 2.3.3. *Assume that $n\mathbb{P}[X_n \in \cdot\,] \stackrel{w^\sharp}{\to} \nu$ in $M_b^\sharp(\overline{\mathbb{C}}_0^+)$ and that $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$. Then, the process $\tilde{M}_n$ weakly converges in $\mathbb{D}([0,+\infty),\mathbb{C}^+)$ as $n \to \infty$ to the superextremal process $\tilde{M}$ defined by*

$$\tilde{M}(u,t) = \sup\{Y_i(t)\mathbf{1}_{[U_i,+\infty)}(u);\ i \geq 1\}, \quad u > 0,\ t \in T, \tag{2.6}$$

where $\sum_{i\geq 1} \delta_{(Y_i,U_i)}$ is a Poisson point process on $\mathbb{C}_0^+ \times [0,+\infty)$ with intensity $\nu \otimes \ell$.

For the proof, we will need the following analoguous of Lemma 2.3.2 in the space-time framework. Let $M^\sharp_{b,p}(\mathbb{C}_0^+\times[0,+\infty))$ be the subset of measures $\mu\in M^\sharp_{(b,p)}(\overline{\mathbb{C}}_0^+\times[0,+\infty))$ such that $\mu((\overline{\mathbb{C}}_0^+\setminus\mathbb{C}_0^+)\times[0,+\infty))=0$ and define

$$\tilde C=\Big\{\mu\in M^\sharp_{(b,p)}(\mathbb{C}_0^+\times[0,+\infty));\ \mu(\overline{\mathbb{C}}_0^+\times\{u\})\leq 1\quad\text{for all } u\geq 0\Big\}.$$

In other words, a measure $\mu=\sum_{i\in I}\delta_{(r_i,s_i,u_i)}$ belongs to $\tilde C$ if and only if $r_i<+\infty$ for all $i\in I$ and the u_i's are pairwise distinct.

Lemma 2.3.4. *The mapping $\tilde\theta: M^\sharp_{(b,p)}(\mathbb{C}_0^+\times[0,+\infty))\to\mathbb{D}([0,+\infty),\mathbb{C}^+)$ defined by $\tilde\theta(0)\equiv 0$ and*

$$\tilde\theta\Big(\sum_{i\in I}\delta_{(r_i,s_i,u_i)}\Big)=\Big((u,t)\mapsto\sup\{r_is_i(t)1_{[u_i,+\infty)}(u);\ i\in I\}\Big)$$

is well defined and continuous on $\tilde C$.

Proof of Lemma 2.3.4:
The proof is similar to the proof of Lemma 2.3.2 and we give only the main lines.

– First we show that $\tilde\theta$ is well defined. Recall that $\overline{\mathbb{C}}_0^+\times[0,+\infty)$ is endowed with the metric

$$\tilde d((r_1,s_1,u_1),(r_2,s_2,u_2))=d((r_1,s_1),(r_2,s_2))+|u_2-u_1|.$$

Let $\mu\in M^\sharp_{(b,p)}(\mathbb{C}_0^+\times[0,+\infty))$. For $\varepsilon>0$ and $M>0$, let $\mu^{(\varepsilon,M)}$ be its restriction to $\bar B^{\varepsilon,M}=[\varepsilon,+\infty]\times\mathbb{S}_{\mathbb{C}^+}\times[0,M]$. Since $\bar B^{\varepsilon,M}$ is bounded, $\mu^{(\varepsilon,M)}$ has only a finite number of atoms and we easily check that $\tilde\theta(\mu^{(\varepsilon,M)})$ belongs to $\mathbb{D}([0,+\infty),\mathbb{C}^+)$. Furthermore, for $u<M$ and $t\in T$,

$$|\tilde\theta(\mu^{(\varepsilon,M)})(u,t)-\tilde\theta(\mu)(u,t)|\leq\varepsilon$$

so that $\tilde\theta(\mu^{(\varepsilon,M)})$ converges uniformly on $[0,M]\times T$ to $\tilde\theta(\mu)$ as $\varepsilon\to 0$. The constant M being arbitrary, this implies that $\tilde\theta(\mu)\in\mathbb{D}([0,+\infty),\mathbb{C}^+)$ and that the application $\tilde\theta$ is well defined. It also holds that

$$\sup_{(u,t)\in[0,M]\times T}|\tilde\theta(\mu^{(\varepsilon,M)})(u,t)-\tilde\theta(\mu)(u,t)|\leq\varepsilon. \tag{2.7}$$

– Next, we show that $\tilde\theta$ is continuous on $\tilde C$.
Let $\{\mu_n=\sum_{i\in I_n}\delta_{x_i^n};\ n\geqslant 1\}$ be a sequence of $M^\sharp_{(b,p)}(\mathbb{C}_0^+\times[0,+\infty))$ converging to $\mu=\sum_{i\in I}\delta_{x_i}\in\tilde C$. For all $\varepsilon>0$ and $M>0$, there exists $\varepsilon'<\varepsilon$, $M'>M$, $N\geq 1$ and some $m\geq 1$, such that

$$\forall n\geqslant N,\quad \mu_n^{(\varepsilon',M')}=\sum_{i=1}^m\delta_{x_i^n}\ \text{ and }\ \mu^{(\varepsilon',M')}=\sum_{i=1}^m\delta_{x_i}$$

with $x_i^n=(r_i^n,s_i^n,u_i^n)$, $x_i=(r_i,s_i,u_i)\in\bar B^{\varepsilon',M'}$ and $x_i^n\to x_i$ as $n\to\infty$. The condition $\mu\in\tilde C$ ensures that the u_i's are pairwise distinct and we can suppose without loss of generality that $u_1 < \cdots < u_m$. For n large enough, we will also have $u_1^n < \cdots < u_m^n$. Define $\delta_{M'}$ the metric associated with the J_1-topology on $\mathbb{D}([0,M'],\mathbb{C}^+)$ by

$$\delta_{M'}(x,y) = \inf_{\lambda} \sup_{(u,t)\in[0,M']\times T} |x(\lambda u,t) - x(u,t)|$$

where the infinimum is taken over the set of non-decreasing homeomorphisms λ of $[0,M']$. Since for large n the u_i's and the u_i^n's are in the same relative order, there exists a non-decreasing homeomorphism $\lambda_{M'}^n$ of $[0,M']$ such that $\lambda_{M'}^n(u_i^n) = u_i$ for all $1 \le i \le m$. We then have

$$\begin{aligned}
&\delta_{M'}(\tilde\theta(\mu_n^{(\varepsilon',M')}),\tilde\theta(\mu^{(\varepsilon',M')}))\\
&\quad\le\quad \delta_{M'}(\tilde\theta(\mu_n)),\tilde\theta(\mu_n^{(\varepsilon',M')})) + \delta_{M'}(\tilde\theta(\mu_n^{(\varepsilon',M')}),\tilde\theta(\mu^{(\varepsilon',M')})) + \delta_{M'}(\tilde\theta(\mu^{(\varepsilon',M')}),\tilde\theta(\mu))\\
&\quad\le\quad \sup_{(u,t)\in[0,M']\times T} |\max\nolimits_{1\le i\le m}(r_i^n s_i^n(\cdot)\mathbf{1}_{[\lambda_{M'}^n u_i^n,+\infty)}(u)) - \max_{1\le i\le m}(r_i s_i(\cdot)\mathbf{1}_{[u_i,+\infty)}(u))|\\
&\quad=\quad \max_{1\le i\le m} \|r_i^n s_i^n - r_i s^i\|\\
&\quad\to\quad 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Hence, for sufficiently large n, $\delta_{M'}(\tilde\theta(\mu_n^{(\varepsilon',M')}),\tilde\theta(\mu^{(\varepsilon',M')})) \le \varepsilon$ and thanks to Equation (2.7), this entails

$$\begin{aligned}
&\delta_{M'}(\tilde\theta(\mu_n)),\tilde\theta(\mu))\\
&\quad\le\quad \delta_{M'}(\tilde\theta(\mu_n)),\tilde\theta(\mu_n^{(\varepsilon',M')})) + \delta_{M'}(\tilde\theta(\mu_n^{(\varepsilon',M')}),\tilde\theta(\mu^{(\varepsilon',M')})) + \delta_{M'}(\tilde\theta(\mu^{(\varepsilon',M')}),\tilde\theta(\mu))\\
&\quad\le\quad 3\varepsilon.
\end{aligned}$$

Since M' is arbitrary large and ε arbitrary small, this proves the convergence $\tilde\theta(\mu_n)$ to $\tilde\theta(\mu)$ in $\mathbb{D}([0,+\infty),\mathbb{C}^+)$.

□

Proof of Theorem 2.3.3:
The proof is very similar to the proof of Theorem 2.3.1. Note that $\tilde M_n = \tilde\theta(\tilde\beta_n)$ where $\tilde\beta_n$ is defined by Equation (2.3). According to Proposition 2.2.1, the empirical measure $\tilde\beta_n$ weakly converges in $M_{(b,p)}(\overline{\mathbb{C}}_0^+ \times [0,+\infty))$ to a Poisson point process $\tilde\Pi_\nu$ with intensity $\nu\otimes\ell$. The assumption $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+)$ and the fact that the Lebesgue measure ℓ has no atoms ensure that $\tilde\Pi_\mu$ lies almost surely in $\tilde C$. From Lemma 2.3.4, the mapping $\tilde\theta$ is continuous on $\tilde C$, so that the continuous mapping Theorem implies $\tilde\theta(\tilde\beta_n) \Rightarrow \tilde\theta(\tilde\Pi_\nu)$ which is equivalent to $\tilde M_n \Rightarrow \tilde M$. □

2.4 Properties of the limit process

In this section, we give some properties of the superextremal process $\tilde M$ defined by Equation (2.6) in Theorem 2.3.3. First, we characterize its finite dimensional distributions. We use vectorial notations : for $l \ge 1$, $\mathbf{t} = (t_1,\ldots,t_l) \in T^l$, $\mathbf{y} = (y_1,\ldots,y_l) \in [0,+\infty)^l$ and $u > 0$, we write $\tilde M(u,\mathbf{t}) \le \mathbf{y}$ if and only if $\tilde M(u,t_i) \le y_i$ for all $i \in \{1,\ldots,l\}$.

Proposition 2.4.1. *For $k \geqslant 1$, $0 = u_0 < u_1 < \cdots < u_k$, $\mathbf{t} \in T^l$ and $\mathbf{y}_1, \ldots, \mathbf{y}_k \in [0, +\infty)^l$, it holds*

$$\mathbb{P}\Big[\tilde{M}(u_1, \mathbf{t}) \leqslant \mathbf{y}_1, ..., \tilde{M}(\mathbf{u}_k, \mathbf{t}) \leqslant \mathbf{y}_k\Big] = \prod_{i=1}^{k} \exp[-(u_j - u_{j-1})\nu(A_j)]$$

with $A_j = \{f \in \mathbb{C}_0^+;\ \exists i \in 1, \ldots, l,\ f(t_i) \geq \min_{k \geq j} y_{k,i}\}$.

Proof of Proposition 2.4.1:
Let $j \in \{1, \ldots, k\}$. Note that $\tilde{M}(u_j, \mathbf{t}) \leqslant \mathbf{y}_j$ if and only if $\tilde{\Pi}_\mu$ doesn't intersect the set

$$B_j = \{(f, u) \in \mathbb{C}_0^+ \times [0, +\infty);\ u \leq u_j \text{ and } \exists i \in \{1, \cdots l\}\ f(t_i) > y_{j,i}\}$$

So, we have

$$\begin{aligned} \mathbb{P}\Big[\tilde{M}(u_1, \mathbf{t}) \leqslant \mathbf{y}_1, ..., \tilde{M}(\mathbf{u}_k, \mathbf{t}) \leqslant \mathbf{y}_k\Big] &= \mathbb{P}\Big[\tilde{\Pi}_\nu \cap (\cup_{j=1}^k B_j) = \emptyset\Big] \\ &= \exp[-(\nu \otimes \ell)(\cup_{j=1}^k B_j)]. \end{aligned}$$

The B_j's are not pairwise disjoint. To compute the measure $(\nu \otimes \ell)(\cup_{j=1}^k B_j)$, we observe that

$$\cup_{j=1}^k B_j = \cup_{j=1}^k ([u_{j-1}, u_j) \times A_j) \cup (\{u_k\} \times A_k)$$

where the sets in the right hand side are pairwise disjoint. From this, we deduce

$$(\nu \otimes \ell)(\cup_{j=1}^k B_j) = \sum_{j=1}^{k} (u_j - u_{j-1})\nu(A_j).$$

This proves the Proposition. □

Next we prove that the process $u \mapsto \tilde{M}(u, \cdot)$ is a $\mathbb{C}^+$-valued homogeneous Markov process. Let $\mathcal{F}_u$ be the σ-algebra generated by $\{\tilde{M}(s, t);\ s \in [0, u], t \in T\}$. The symbol $\vee$ stands for poinwise maximum.

Proposition 2.4.2. *Let $u \geq 0$. The conditional distribution of $(\tilde{M}(u+h, \cdot))_{h \geq 0}$ given $\mathcal{F}_u$ is equal to the distribution of $(\tilde{M}(u, \cdot) \vee \tilde{M}'(h, \cdot))_{h \geq 0}$ where $\tilde{M}'$ is an independent copy of $\tilde{M}$.*

In some sense, this Proposition states that the process $\tilde{M}$ has "independent and stationary increments" with respect to the maximum : for $0 = u_0 < u_1 < \cdots < u_k$, the distribution of $(\tilde{M}(u_i, \cdot))_{1 \leq i \leq k}$ is equal to the distribution of

$$(\vee_{j=1}^i \tilde{M}^j(u_j - u_{j-1}, \cdot))_{1 \leq i \leq k},$$

where $\tilde{M}^1, \ldots, \tilde{M}^k$ are i.i.d. copies of $\tilde{M}$. This property is similar to the property of independence and stationarity of increments (with respect to the addition) of Lévy processes. For a fixed point $t \in T$, the process $\{M(u, t), \quad u \geq 0\}$ is known as an extremal process (see Proposition 4.7 of Resnick [62]).

Proof of Proposition 2.4.2:
Consider the decomposition $\tilde{\Pi}_\mu = \tilde{\Pi}_\nu^{[0,u]} \cup \tilde{\Pi}_\nu^{(u,\infty)}$ where

$$\tilde{\Pi}_\nu^{[0,u]} = \tilde{\Pi}_\nu \cap (\mathbb{C}_0^+ \times [0,u]) \quad \text{and} \quad \tilde{\Pi}_\nu^{(u,\infty)} = \tilde{\Pi}_\nu \cap (\mathbb{C}_0^+ \times (u,\infty)).$$

By the independence properties of the Poisson point process, $\tilde{\Pi}_\mu^{[0,u]}$ and $\tilde{\Pi}_\mu^{(u,\infty)}$ are independent. Furthermore,

$$\begin{aligned} \tilde{M}(u+h,\cdot) &= \sup\{rs(\cdot)1_{[v,+\infty)}(u+h);\ (r,s,v) \in \tilde{\Pi}_\nu^{[0,u]}\} \\ &\qquad \vee \ \sup\{rs(\cdot)1_{[v,+\infty)}(u+h);\ (r,s,v) \in \tilde{\Pi}_\nu^{(u,\infty)}\} \end{aligned}$$

and the two terms in the r.h.s. are independent. It is easily seen that the first term is equal to $\tilde{M}(u,\cdot)$, and that the invariance of the Lebesgue measure ℓ implies that the second term has the same distribution as $\tilde{M}(h,\cdot)$. This proves the Proposition. □

In the particular case when the measure ν is homogeneous of order $-\alpha < 0$, the process $\tilde{M}$ enjoys further interesting properties.

Proposition 2.4.3. *Suppose ν is homogeneous with index $-\alpha < 0$. Then :*
- *$\tilde{M}$ is max-stable with index $\alpha > 0$, i.e. if $\tilde{M}^1, \ldots, \tilde{M}^n$ are independent copies of $\tilde{M}$, the maximum $\vee_{i=1}^n \tilde{M}^i$ has the same law as $n^{\frac{1}{\alpha}}\tilde{M}$.*
- *$\tilde{M}$ is self-similar with index $\frac{1}{\alpha}$, i.e. for all $c > 0$, the rescaled process $(\tilde{M}(cu,\cdot))_{u\geq 0}$ has the same distribution as $(c^{1/\alpha}\tilde{M}(cu,\cdot))_{u\geq 0}$*

Proof of Proposition 2.4.3:
We check that the the two processes have the same finite dimensional distributions. To this aim, we use Proposition 2.4.1 and the notations of the proposition.
- Let $\tilde{M}^1, \ldots, \tilde{M}^n$ be independent copies of $\tilde{M}$. By Proposition 2.4.1 and the homogeneity of μ,

$$\begin{aligned} \mathbb{P}\Big[\vee_{i=1}^n \tilde{M}^i(u_j,\mathbf{t}) \leqslant \mathbf{y}_j,\ \forall j \in \{1,\ldots,k\}\Big] &= \mathbb{P}\Big[\tilde{M}(u_j,\mathbf{t}) \leqslant \mathbf{y}_j,\ \forall j \in \{1,\ldots,k\}\Big]^n \\ &= \prod_{i=1}^k \exp[-(u_j-u_{j-1})\nu(A_j)]^n \\ &= \prod_{i=1}^k \exp[-(u_j-u_{j-1})\nu(n^{-1/\alpha}A_j)] \\ &= \mathbb{P}\Big[n^{1/\alpha}\tilde{M}(u_j,\mathbf{t}) \leqslant \mathbf{y}_j,\ \forall j \in \{1,\ldots,k\}\Big]. \end{aligned}$$

This proves the max-stability.

– Similarly, the self-similarity is proven as follows :

$$\begin{aligned}\mathbb{P}\Big[\tilde{M}(cu_j,\mathbf{t}) \leqslant \mathbf{y}_j,\ \forall j \in \{1,\ldots,k\}\Big] &= \prod_{i=1}^{k} \exp[-c(u_j - u_{j-1})\nu(A_j)] \\ &= \prod_{i=1}^{k} \exp[-(u_j - u_{j-1})\nu(c^{-1/\alpha}A_j)] \\ &= \mathbb{P}\Big[c^{1/\alpha}\tilde{M}(u_j,\mathbf{t}) \leqslant \mathbf{y}_j,\ \forall j \in \{1,\ldots,k\}\Big].\end{aligned}$$

□

2.5 Further order statistics

The point process approach for extremes is powerful : the convergence of the empirical measure β_n entails not only the convergence of the maxima M_n but also of all the order statistics, and a similar result hold for the space-time version of these processes.

For $1 \leq r \leq n$ and $t \in T$, define $M_n^r(t)$ as the r-th largest value among $X_{1n}(t),\ldots,X_{nn}(t)$. Note $M_n^1(t) = M_n(t)$ is simply the maximum. For $r > n$, we use the convention $M_n^r(t) = 0$. The process $M_n^r = (M_n^r(t))_{t\in T}$ is refered to as the r-th order statistic of the sample $\{X_{1n},\ldots,X_{1n}\}$. Similarly, for $r \geq 1$, $t \in T$ and $u \geq 0$ define $\tilde{M}_n^r(u,t)$ as the r-th largest value among $X_{1n}(t),\ldots,X_{[nu]n}(t)$ with the convention $\tilde{M}_n^r(u,t) = 0$ if $r > [nu]$. An alternating definition in terms of the empirical measure $\tilde{\beta}_n$ is given by

$$\tilde{M}_n^r(u,t) = \sup\{y \geq 0;\ \tilde{\beta}_n(B_{u,t}^y) \geq r\}$$

with $B_{u,t}^y = \{(f,v) \in \mathbb{C}_0^+ \times [0,+\infty); f(t) \geq y, v \leq u\}$ and the convention that the supremum over an empty set is equal to zero.

With these notations, we can strengthen Theorem 2.3.3 as follows :

Theorem 2.5.1. *Assume that $n\mathbb{P}[X_n \in \cdot\,] \overset{w^\sharp}{\to} \nu$ in $M_b^\sharp(\overline{\mathbb{C}}_0^+)$ and that $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$. Then, for each $r \geq 1$, the joint order statistics $(\tilde{M}_n^1,\ldots,\tilde{M}_n^r)$ weakly converges in $\mathbb{D}([0,+\infty),(\mathbb{C}^+)^r)$ as $n \to \infty$ to $(\tilde{M}^1,\ldots,\tilde{M}^r)$ defined by*

$$\tilde{M}^j(u,t) = \sup\{y \geq 0;\ \tilde{\Pi}_\nu \cap B_{u,t}^y \geq r\}, \quad 1 \leq j \leq r,\ u \geq 0,\ t \in T,$$

with $\tilde{\Pi}_\nu$ a Poisson point process on $\mathbb{C}_0^+ \times [0,+\infty)$ with intensity $\nu \otimes \ell$.

The limit process $\tilde{M}^j$ corresponds to the r-th space-time order statistic associated to the point process $\tilde{\Pi}_\nu$.

Proof of Theorem 2.5.1:

Using once again Proposition 2.2.1 and the continuous mapping Theorem, it is enough to prove the following generalization of Lemma 2.3.4 : the mapping

$$\tilde{\theta}^r : M_{(b,p)}^\sharp(\mathbb{C}_0^+ \times [0,+\infty)) \to \mathbb{D}([0,+\infty),(\mathbb{C}^+)^r)$$

defined by $\tilde{\theta}(0) \equiv 0$ and

$$\tilde{\theta}(\beta)(u,t) = (\sup\{y \geq 0;\ \tilde{\beta}(B^y_{u,t}) \geq j\})_{1\leq j\leq r}$$

is well defined and continuous on $\tilde{C}$. The proof is very similar to the proof of Lemma 2.3.4 and the details are omitted for the sake of brevity. □

2.6 Gaussian case

Gaussian processes can be very useful for modelisation in EVT, Kabluchko, Schlather and de Haan in [51] have extended its use for the construction of Brown-Resnick processes. Thereafter the work of Kabluchko [49] on extreme of gaussian process inspired us to use Gaussian processes, to create processes (log-normal processes) verifying the condition given in section 3 to construct partial maxima processes which converge to a superextremal process.

So let $\{Z_n, n \in \mathbb{N}\}$ be a sequence of continuous zero-mean Gaussian, unit-variance processes with covariance function $r_n(t_1,t_2) = \mathbb{E}[Z_n(t_1)Z_n(t_2)]$, $t_1,t_2 \in T$. Here (T,d) is a compact metric space verifying :

$$\int_0^1 (\log N(\varepsilon))^{1/2} d\varepsilon < \infty$$

where $N(\varepsilon)$ is the smallest number of balls of raduis ε needed to cover T. Let define following processes :

$$\{Y_n(t) = b_n(Z_n(t) - b_n), \quad t \in T\},$$

$$\text{with } b_n = (2\log n)^{1/2} - (2\log n)^{-1/2}((1/2)\log\log n + \log(2\sqrt{\pi})),$$

$$\text{and the log-normal process} \{X_n(t) = \exp Y_n(t), \quad t \in T\}.$$

Theorem 2.6.1. *Fix $t_0 \in T$. Suppose that :*

i) Uniformly in $t_1, t_2 \in T$:

$$\Gamma(t_1,t_2) = \lim_{n\to\infty} 4(1 - r_n(t_1,t_2))\log n \in [0,\infty). \tag{2.8}$$

ii) For all $t_1,t_2 \in T$, $\exists \alpha > 0, C_1 > 0$:

$$\sup_{n\geqslant 1} 2(1 - r_n(t_1,t_2))\log n \leqslant C_1 d(t_1,t_2). \tag{2.9}$$

Then :

$$\lim_{n\to\infty} n\mathbb{P}[X_n \in \cdot\,] \xrightarrow{w^\sharp} \nu(\cdot) \text{ in } M_b^\sharp(\mathbb{C}),$$

$$\text{where } \nu(A) = \int_0^\infty \mathbb{P}[we^{W(\cdot)-\frac{1}{2}\Gamma(t_0,\cdot)} \in A]\frac{1}{w^2}dw, \quad A \text{ is a bounded set of } \mathbb{C},$$

$\{W(t), t \in \mathbb{R}^d\}$ is a Gaussian process with incremental variance Γ such that $W(t_0) = 0$.

Let Y_n^w be the process Y_n conditioned by $Y_n(t_0) = w$, with $\mathbb{E}[Y_n^w(t)] = \mu_n^w(t)$. We have :

$$\mu_n^w(t) = wr_n(t, t_0) + b_n^2(r_n(t, t_0) - 1)$$

$$Cov[Y_n^w(t_1), Y_n^w(t_2)] = b_n^2 \left(r_n(t_1, t_2) - r_n(t_1, t_0)r_n(t_2, t_0)\right).$$

Lemma 2.6.2. *If the criterions of the theorem hold, then the family $Y_n^w - \mu_n^w$, $w \in \mathbb{R}, n \in \mathbb{N}$ is tight un $\mathbb{C}(K)$. Fix w , the family $(Y_n^w)_{n\in\mathbb{N}}$ is tight.*

Proof of Lemma 2.6.2:
This lemma has already been proved, when the criterions of the theorem hold, by Kabluchko (see demonstration of Theoreme 6 of [49]).

□

Lemma 2.6.3. *If the conditions of 2.6.1 hold, then :*

$$\{Y_n^w(t), t \in \mathbb{R}^d\} \xrightarrow{f.d} \{w + W(t) - \frac{1}{2}\Gamma(t, t_0), t \in T\}.$$

Proof of Lemma 2.6.3:
We have :

$$\mu_n^w(t) = wr_n(t, t_0) + b_n^2(r_n(t, t_0) - 1)$$

$$Cov[Y_n^w(t_1), Y_n^w(t_2)] = b_n^2(r_n(t_1, t_2) - r_n(t_1, t_0)r_n(t_2, t_0)).$$

Since for every $t_1, t_2 \in T$, $\Gamma(t_1, t_2) < \infty$, Hence :

$$\lim_{n\to\infty} \mu_n^w(t) = w - \frac{1}{2}\Gamma(t, t_0)$$

$$\lim_{n\to\infty} Cov[Y_n^w(t_1), Y_n^w(t_2)] = \frac{1}{2}(\Gamma(t_1, t_0) + \Gamma(t_2, t_0) - \Gamma(t_1, t_2)).$$

Therefore :

$$\{Y_n^w(t), t \in T\} \xrightarrow{f.d} \{w + W(t) - \frac{1}{2}\Gamma(t, t_0), t \in T\}.$$

□

Proof of Theorem 2.6.1:

This proof is inspired by the work of Kabluchko, Schlather and de Haan (see proof of theorem 17 of [51]), but here, to prove the theorem we will use the convergence criterions given by Hult et Lindskog (see theorem 4.4 of [47]) :

i) $\sup_n n\mathbb{P}[\sup_{t\in K} X_n(t) > r] < \infty$.

ii) $\lim_{\delta\searrow 0} \sup_n n\mathbb{P}[\omega_{X_n}(\delta) \geqslant \varepsilon] = 0$.

iii) Convergence to ν in the sense of finite-dimensional .

• Proof of i).
For $\tilde{r} = \ln r$, We have :

$$\begin{aligned} n\mathbb{P}[\sup_{t\in T} X_n(t) > r] &= n\mathbb{P}[\sup_{t\in K} Y_n(t) > \tilde{r}] \\ &= \frac{n}{\sqrt{2\pi} b_n e^{b_n^2/2}} \int_{\mathbb{R}} e^{-w-w^2/2b_n^2} \mathbb{P}[\sup_{t\in T} Y_n^w(t) > \tilde{r}] dw \\ &\leqslant 2 \int_{\mathbb{R}} e^{-w+w^2/2b_n^2} \mathbb{P}[\sup_{t\in T} Y_n^w(t) > \tilde{r}] dw \ . \end{aligned}$$

For every $c_1 \in \mathbb{R}$, there exists some $h_1 > 0$ such that

$$\int_{c_1}^{+\infty} e^{-w-w^2/2b_n^2} \mathbb{P}[\sup_{t\in T} Y_n^w(t) > \tilde{r}] dw \leqslant \int_{c_1}^{+\infty} e^{-w} dw = h_1 < \infty$$

By Lemma 2.6.2, there exists $c_2 > 0$ such that

$$\mathbb{P}[\sup_{t\in T} (Y_n^w(t) - \mu_n^w(t)) > c_2] < \frac{1}{2}.$$

For some c_3, c_4 we have

$$\sup_{t\in T} \mu_n^w(t) \leqslant w + c_3, \text{ and } \sup_{t\in T} VarY_n^w(t) \leqslant c_4^2.$$

Applying Borell's inequality, we obtain

$$\mathbb{P}[\sup_{t\in T} Y_n^w(t_j) > \tilde{r}] < 2\psi(\frac{\tilde{r} - w - c_2 - c_3}{c_4}).$$

Where ψ is the tail of the standard Gaussian distribution. We have

$$\psi(\frac{\tilde{r} - w - c_2 - c_3}{c_4}) \leqslant e^{-(\frac{\tilde{r}-w-c_2-c_3}{c_4})^2}$$

Choosing w sufficiently small, we obtain :

$$\mathbb{P}[\sup_{t\in T} Y_n^w(t_j) > \tilde{r}] < 2e^{-\frac{w^2}{32c_4}} \tag{2.10}$$

So, choosing c_1 sufficiently small, there exists some $h_2 > 0$ such that

$$\int_{-\infty}^{c_1} e^{-w-w^2/2b_n^2} \mathbb{P}[\sup_{t\in K} Y_n^w(t_j) > \tilde{r} - w] dw < \int_{-\infty}^{c_1} e^{-w-\frac{w^2}{32c_4}} dw = h_2 < \infty.$$

Therefore, there exists some h, we can choose here $h = h_1 + h_2$, such that :

$$\sup_n n\mathbb{P}[\sup_{t\in T} X_n(t) > r] < h < \infty$$

• Proof of ii).

–For all $\delta > 0$, we have

$$\omega_{X_n}(\delta) = \omega_{e^{Y_n}}(\delta)$$

Hence

$$\begin{aligned}\sup_n n\mathbb{P}[\omega_{X_n}(\delta) \geqslant \varepsilon] &= \sup_n \frac{n}{\sqrt{2\pi} b_n e^{b_n^2/2}} \int_{\mathbb{R}} e^{-w-\frac{w^2}{2b_n^2}} \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geqslant \varepsilon] dw \\ &\leqslant 2 \int_{\mathbb{R}} e^{-w} \sup_n \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geqslant \varepsilon] dw \ .\end{aligned}$$

By tightness of the sequence $\{e^{Y_n^w}, \quad n \geqslant 1\}$, for every fixed w, we have :

$$\lim_{\delta \searrow 0} \sup_n \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geqslant \varepsilon] = 0$$

therefore

$$\lim_{\delta \searrow 0} e^{-w} \sup_n \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geqslant \varepsilon] = 0$$

–For all $0 < \delta < \delta_0$, $c > 0$, we have

$$e^{-w} \sup_n \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geqslant \varepsilon] \leqslant \begin{cases} e^{-w} \text{ if } w > -c \\ e^{-w} \sup_n \mathbb{P}[\omega_{e^{Y_n^w}}(\delta_0) \geqslant \varepsilon] \text{ if } w < -c \end{cases}$$

$$\int_{-c}^{+\infty} e^{-w} dw = e^c < \infty \text{ and } \omega_{e^{Y_n^w}}(\delta_0) \leqslant \sup(e^{Y_n^w})$$

so, we have

$$\mathbb{P}[\omega_{e^{Y_n^w}}(\delta_0) \geqslant \varepsilon] \leqslant \mathbb{P}[\sup(Y_n^w) \geqslant \ln \varepsilon] \text{ si } w < -c$$

By (2.10), for some c sufficiently large, $\forall C$, there exists some h such that

$$\int_{-\infty}^{-c} e^{-w} \sup_n \mathbb{P}[\sup(Y_n^w) \geqslant C] dw < h < \infty.$$

Hence, for $0 < \delta < \delta_0$, we have :

$$\int_{\mathbb{R}} e^{-w} \sup_n \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geqslant \varepsilon] dw < h + e^c < \infty.$$

therefore, we can use the dominated convergence theorem

$$\lim_{\delta \searrow 0} \sup_n n\mathbb{P}[\omega_{X_n}(\delta) \geqslant \varepsilon] = 0.$$

• Proof of iii).

Let $J \subset \mathbb{N}$ be a finite subset with $|J| = k$, $\{t_j, \quad j \in J\}$ be a sequence of T and $\{y_j, \quad j \in J\}$ be a sequence of $\mathbb{R}$. We have

$$\sqrt{2\pi} b_n e^{b_n^2/2} \sim n \text{ and } f_{Y_n(t_0)}(w) = \frac{1}{\sqrt{2\pi} b_n} e^{-(w+b_n^2)^2/2b_n^2}$$

So, for $y_j = \ln x_j$ we obtain

$$\begin{aligned} n\mathbb{P}[\exists j \in J : X_n(t_j) > x_j] &= n\mathbb{P}[\exists j \in J : Y_n(t_j) > y_j] \\ &= \frac{n}{\sqrt{2\pi} b_n} \int_{\mathbb{R}} e^{-\frac{(w+b_n^2)^2}{2b_n^2}} \mathbb{P}[\exists j \in J : Y_n(t_j) > y_j | Y_n(t_0) = w] dw \\ &= \frac{n}{\sqrt{2\pi} b_n e^{b_n^2/2}} \int_{\mathbb{R}} e^{-w-\frac{w^2}{2b_n^2}} \mathbb{P}[\exists j \in J : Y_n^w(t_j) > y_j] dw \\ &\sim \int_{\mathbb{R}} e^{-w-\frac{w^2}{2b_n^2}} \mathbb{P}[\exists j \in J : Y_n^w(t_j) > y_j] dw \end{aligned}$$

And

$$\begin{aligned} \mathbb{P}[\exists j \in J : Y_n^w(t_j) > y_j] &\leqslant \sum_{j\in J} \mathbb{P}[Y_n^w(t_j) > y_j] \\ &\leqslant \sum_{j\in J} \mathbb{P}\left[\sup_{t\in T} Y_n^w(t) > y_j\right]. \end{aligned}$$

Since there existe some h such that for all n, j :

$$\int_{\mathbb{R}} e^{-w-\frac{w^2}{2b_n^2}} \mathbb{P}\left[\sup_{t\in T} Y_n^w(t) > y_j\right] dw < h < \infty,$$

so we obtain

$$\int_{\mathbb{R}} e^{-w-\frac{w^2}{2b_n^2}} \mathbb{P}[\exists j \in J : Y_n^w(t_j) > y_j] dw < kh < \infty.$$

Furthermore since

$$\lim_{n\to\infty} e^{-w-\frac{w^2}{2b_n^2}} \mathbb{P}[\exists j \in J : Y_n^w(t_j) > y_j] = e^{-w} \mathbb{P}[\exists j \in J : w + W(t_j) - \frac{1}{2}\Gamma(t_j, t_0) > y_j]$$

by applying dominated convergence theorem, we obtain

$$\lim_{n\to\infty} n\mathbb{P}[\exists j \in J : X_n(t_j) > x_j] = \int_{\mathbb{R}} e^{-w} \mathbb{P}[\exists j \in J : w + W(t_j) - \frac{1}{2}\Gamma(t_j, t_0) > y_j] dw.$$

We may write the above as

$$\lim_{n\to\infty} n\mathbb{P}[\exists j \in J : X_n(t_j) > x_j] = \int_0^\infty \mathbb{P}[\exists j \in J : w e^{W(t_j)-\frac{1}{2}\Gamma(t_j,t_0)} > x_j] \frac{1}{w^2} dw.$$

Therefore $n\mathbb{P}[X_n \in \cdot\,]$ converges to ν in finite-dimensional sense .

The three conditions of the convergence criterion hold, so we have

$$\lim_{n\to\infty} n\mathbb{P}[X_n \in \cdot\,] \xrightarrow{w^\#} \nu(\cdot) \text{ in } \bar{\mathbb{C}}_0^+.$$

□

Remark 2.6.4. *As in Kabluchko [49], Theorem 2.6.1 can easily be generalised to deal with non reduced process (zero-mean Gaussian processes) under a suitable scaling assumption of the covariance function.*

Chapitre 3

Regular conditional distributions of max infinitely divisible processes

Abstract

This paper is devoted to the *prediction problem* in extreme value theory. Our main result is an explicit expression of the *regular conditional distribution* of a max-stable (or max-infinitely divisible) process $\{\eta(t)\}_{t\in T}$ given observations $\{\eta(t_i) = y_i,\ 1 \leq i \leq k\}$. Our starting point is the point process representation of max-infinitely divisible processes by Giné et al. [44]. We carefully analyze the structure of the underlying point process, introduce the notions of *extremal function*, *sub-extremal function* and *hitting scenario* associated to the constraints and derive the associated distributions. This allows us to explicit the conditional distribution as a mixture over all hitting scenarios compatible with the conditioning constraints. This formula extends a recent result Wang and Stoev [79] dealing with the case of spectrally discrete max-stable random fields. This paper offers new tools and perspective for prediction in extreme value theory together with numerous potential applications.

3.1 Introduction

3.1.1 Motivations

Max-stable random fields turn out to be fundamental models for spatial extremes since they arise as the the limit of rescaled maxima. More precisely, consider the component-wise maxima

$$\eta_n(t) = \max_{1\leq i\leq n} X_i(t), \quad t \in T,$$

of independent and identically (i.i.d.) random fields $\{X_i(t)\}_{t\in T}$, $i \geq 1$. If the random field $\eta_n = (\eta_n(t))_{t\in T}$ converges in distribution, as $n \to \infty$, under suitable affine normalization, then its limit $\eta = \{\eta(t)\}_{t\in T}$ is necessarily max-stable (see e.g. [31, 62]). Therefore, max-stable random fields play a central role in extreme value theory, just like Gaussian random fields do in the classical statistical theory based on the Central Limit Theorem.

Since the pioneer works by Fisher and Typett [40] and Gnedenko [45], the univariate theory of extremes is now well established with extensive studies on models, domains of attraction, parameter estimations, *etc.* (see e.g. de Haan and Fereira [31] and the references therein). The last decades have seen the quick development of multivariate and spatial extreme value theory : the emphasis is put on the characterization, modeling and estimation of the dependence structure of multivariate extremes. Among many others, the reader should refer to the excellent monographs [6, 31, 38, 62] and the reference therein.

In this framework, the *prediction problem* arises as an important and long-standing challenge in extreme value theory. Suppose that one already has a suitable max-stable model for the dependence structure of a random field $\eta = \{\eta(t)\}_{t\in T}$ and that the field is observed at some locations $t_1, \ldots, t_k \in T$. How can we take benefit from these observations and predict the random field η at other locations ? We are naturally lead to consider the *conditional distribution* of $(\eta(t))_{t\in T}$ given the observations $\{\eta(t_i) = y_i,\ 1 \leq i \leq k\}$. A formal definition of the notion of regular conditional distribution is deferred to the Appendix 3.A.2.

In the classical Gaussian framework, i.e., if η is a Gaussian random field, it is well known that the corresponding conditional distribution remains Gaussian and simple formulas give the conditional mean and covariance structure. This theory is strongly linked with the theory of Hilbert spaces : the conditional expectation, for example, can be obtained as the L^2-projection of the random field η onto a suitable Gaussian subspace. In extreme value theory, the prediction problem turns out to be difficult. A first approach by Davis and Resnick [23, 24] is based on a L^1-metric between max-stable variables and on a kind of projection onto max-stable spaces. To some extent, this work mimics the corresponding L^2-theory for Gaussian spaces. However, unlike the Gaussian case, there is no clear relationship between the predictor obtained by projection onto the max-stable space generated by the variables $\{\eta(t_i),\ 1 \leq i \leq k\}$ and the conditional distributions of η with respect to these variables. A first major contribution to the conditional distribution problem is the work by Wang and Stoev [79]. The authors consider max-linear random fields, a special class of max-stable random fields with discrete spectral measure, and give an exact expression of the conditional distributions as well as efficient algorithms. The max-linear structure plays an essential role in their work and provides major simplifications since in this case η admits the simple representation

$$\eta(t) = \bigvee_{j=1}^{q} Z_j f_j(t), \quad t \in T,$$

where the symbol $\bigvee$ denotes the maximum, $f_1, \ldots, f_q$ are deterministic functions and $Z_1, \ldots, Z_q$ are independent and identically distributed (i.i.d.) random variables with unit Fréchet distribution. The authors determine the conditional distributions of $(Z_j)_{1\leq j\leq q}$ given observations $\{\eta(t_i) = y_i,\ 1 \leq i \leq k\}$. Their result relies on the important notion of *hitting scenario* defined as the subset of indices $j \in [\![1, q]\!]$ such that $\eta(t_i) = Z_j f(t_i)$ for some $i \in [\![1, k]\!]$, where, for $n \geq 1$, we note $[\![1, n]\!] = \{1, \ldots, n\}$. The conditional distribution of $(Z_j)_{1\leq j\leq q}$ is expressed as a mixture over all admissible hitting scenarios with minimal rank. The purpose of the present paper is to propose a general theoretical framework for conditional distributions in extreme value theory, covering not only the whole class of sample continuous max-stable random fields but also the class of sample continuous max-infinitely divisible (max-i.d.) random fields (see Balkema and Resnick [5]).

Our starting point is the general representation by Giné, Hahn and Vatan [44] of max-i.d. sample continuous random fields. It is possible to construct a Poisson random measure $\Phi = \sum_{i=1}^{N} \delta_{\phi_i}$ on the space of continuous functions on T such that

$$\eta(t) \stackrel{\mathcal{L}}{=} \bigvee_{i=1}^{N} \phi_i(t), \quad t \in T.$$

Here the random variable N is equal to the total mass of Φ that may be finite or infinite and $\stackrel{\mathcal{L}}{=}$ stands for equality of probability laws (see Theorem 3.1.1 below for a precise statement). We denote by $[\Phi] = \{\phi_i,\ 1 \leq i \leq N\}$ the set of atoms of Φ. Clearly, $\phi(t) \leq \eta(t)$ for all $t \in T$ and $\phi \in [\Phi]$. The observations $\{\eta(t_i) = y_i,\ 1 \leq i \leq k\}$ naturally lead to consider extremal points : a function $\phi \in [\Phi]$ is called *extremal* if $\phi(t_i) = \eta(t_i)$ for some $i \in [\![1, k]\!]$, otherwise it is called *sub-extremal*. We show that under some mild condition, one can define a random partition $\Theta = (\theta_1, \ldots, \theta_\ell)$ of $\{t_1, \ldots, t_k\}$ and extremal functions $\varphi_1^+, \ldots, \varphi_\ell^+$ such that the point t_i belongs to the component θ_j if and only if $\varphi_j^+(t_i) = \eta(t_i)$. Using the terminology of Wang and Stoev [79], we call *hitting scenario* a partition of $\{t_1, \cdots, t_k\}$ that reflects the way how the extremal functions $\varphi_1^+, \ldots, \varphi_\ell^+$ hit the constraints $\varphi_j^+(t_i) \leq \eta(t_i)$, $1 \leq i \leq k$. The main results of this paper are Theorems 3.3.2 and 3.3.3, where the conditional distribution of η given $\{\eta(t_i) = y_i,\ 1 \leq i \leq k\}$ is expressed as a *mixture* over all possible hitting scenarios.

The paper is structured as follows. In Section 2, the distribution of extremal and sub-extremal points is analyzed and a characterization of the hitting scenario distribution is given. In Section 3, we focus on conditional distributions : we compute the conditional distribution of the hitting scenario and extremal functions and then derive the conditional distribution of η. Section 4 is devoted to examples : we specify our results in the simple case of a single conditioning point and consider max-stable models. The proofs are collected in Section 5 and some technical details are postponed to an appendix.

3.1.2 Preliminary on max-i.d. processes

Let T be a compact metric space and $\mathcal{C} = \mathcal{C}(T, \mathbb{R})$ be the space of continuous functions on T endowed with the sup norm

$$\|f\| = \sup_{t \in T} |f(t)|, \quad f \in \mathcal{C}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random process $\eta = \{\eta(t)\}_{t \in T}$ is said to be max-i.d. on $\mathcal{C}$ if η has a version with continuous sample path and if, for each $n \geq 1$, there exists $\{\eta_{ni},\ 1 \leq i \leq n\}$ independent and identically distributed (i.i.d.) sample continuous random fields on T such that

$$\eta \stackrel{\mathcal{L}}{=} \bigvee_{i=1}^{n} \eta_{ni},$$

where $\vee$ denotes pointwise maximum.

Giné, Hahn and Vatan (see [44] Theorem 2.4) give a representation of such processes in terms of Poisson random measure. For any function f on T and set $A \subset T$, we note $f(A) = \sup_{t \in A} f(t)$.

Theorem 3.1.1. *(Giné, Hahn and Vatan [44])*
Let h be the vertex function of a sample continuous max-i.d. process η defined by

$$h(t) = \sup\{x \in \mathbb{R};\ \mathbb{P}(\eta(t) \geq x) = 1\} \in [-\infty, \infty), \quad t \in T,$$

and define $\mathcal{C}_h = \{f \in \mathcal{C};\ f \neq h, f \geq h\}$.
Under the condition that the vertex function h is continuous, there exists a locally-finite Borel measure μ on $\mathcal{C}_h$, such that if Φ is a Poisson random measure Φ on $\mathcal{C}_h$ with intensity measure μ, then

$$\{\eta(t)\}_{t \in T} \stackrel{\mathcal{L}}{=} \{\sup\{h(t), \phi(t); \phi \in [\Phi]\}\}_{t \in T} \tag{3.1}$$

where $[\Phi]$ denotes the set of atoms of Φ.
Furthermore, the following relations hold :

$$h(K) = \sup\{x \in \mathbb{R};\ \mathbb{P}(\eta(K) \geq x) = 1\}, \quad K \subset T \text{ closed}, \tag{3.2}$$

and

$$\mathbb{P}\left[\eta(K_i) < x_i,\ 1 \leq i \leq n\right] = \exp[-\mu\left(\cup_{i=1}^n \{f \in \mathcal{C}_h;\ f(K_i) \geq x_i\}\right)], \tag{3.3}$$

where $n \in \mathbb{N}$, $K_i \subset T$ closed and $x_i > h(K_i)$, $1 \leq i \leq n$.

Theorem 3.1.1 provides an almost complete description of max-i.d. continuous random processes, the only restriction being the continuity of the vertex function. Clearly, the distribution of η is completely characterized by the vertex function h and the so called *exponent measure* μ. The random process $e^\eta - e^h$ is continuous and max-i.d. and its vertex function is identically equal to 0. Since the conditional distribution of η is easily deduced from that of $e^\eta - e^h$, we can assume without loss of generality that $h \equiv 0$; the corresponding set $\mathcal{C}_0$ is the space of non negative and non null continuous functions on T.

We need some more notations from point process theory (see Daley and Vere-Jones [20, 21]). It will be convenient to introduce a measurable enumeration of the atoms of Φ (see [21] Lemma 9.1.XIII). The total mass of Φ is noted $N = \Phi(\mathcal{C}_0)$. If $\mu(\mathcal{C}_0) < \infty$, N has a Poisson distribution with mean $\mu(\mathcal{C}_0)$, otherwise $N = +\infty$ almost surely (a.s.). One can construct $\mathcal{C}_0$-valued random variables $(\phi_i)_{i \geq 1}$ such that $\Phi = \sum_{i=1}^N \delta_{\phi_i}$.

Let $M_p(\mathcal{C}_0)$ be the space of point measures $M = \sum_{i \in I} \delta_{f_i}$ on $\mathcal{C}_0$ such that

$$\{f_i \in \mathcal{C}_0 \colon \|f_i\| > \varepsilon\} \text{ is finite for all } \varepsilon > 0.$$

We endow $M_p(\mathcal{C}_0)$ with the σ-algebra $\mathcal{M}_p$ generated by the applications

$$M \mapsto M(A), \quad A \subset \mathcal{C}_0 \text{ Borel set .}$$

For $M \in M_p(\mathcal{C}_0)$, let $[M] = \{f_i,\ i \in I\}$ be the set of countable atoms of M. If M is non null, then for all $t \in T$, the set $\{f(t); f \in [M]\}$ is non empty and has finitely many points in $(\varepsilon, +\infty)$

for all $\varepsilon > 0$ so that the maximum $\max\{f(t); f \in [M]\}$ is reached. Furthermore by considering restrictions of the measure M to sets $\{f \in \mathcal{C}_0;\ \|f\| > \varepsilon\}$ and using uniform convergence, it is easy to show that the mapping

$$\max(M)\colon \begin{cases} T \to [0, +\infty) \\ t \mapsto \max\{f(t); f \in [M]\} \end{cases}$$

is continuous with the convention that $\max(M) \equiv 0$ if $M = 0$.

In Theorem 3.1.1 (with $h \equiv 0$), Equation (3.3) implies that the exponent measure μ satisfies, for all $\varepsilon > 0$,

$$\mu(\{f \in \mathcal{C}_0;\ \|f\| > \varepsilon\}) < \infty. \tag{3.4}$$

Consequently, we have $\Phi \in M_p(\mathcal{C}_0)$ almost surely and $\eta \stackrel{\mathcal{L}}{=} \max(\Phi)$.

An illustration of Theorem 3.1.1 is given in Figure 3.1 with a representation of the Poisson point measure Φ and of the corresponding maximum process $\eta = \max(\Phi)$ in the moving maximum max-stable model based on the Gaussian density function.

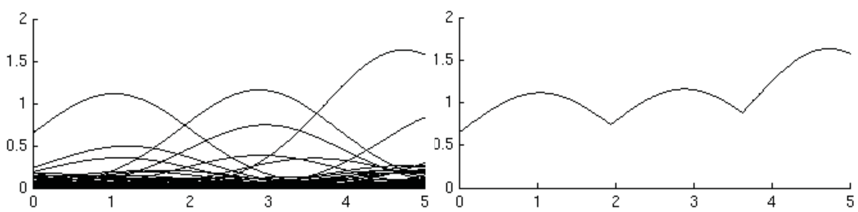

FIGURE 3.1 – A representation of the point process Φ (left) and of the associated maximum process $\eta = \max(\Phi)$ (right) in the moving maximum max-stable model based on the gaussian density function. Here $T = [0, 5]$.

3.2 Extremal points and related distributions

In the sequel, η denotes a sample continuous max-i.d. random process with vertex function $h \equiv 0$ and exponent measure μ on $\mathcal{C}_0$. On the same probability space, we suppose that a $M_p(\mathcal{C}_0)$-valued Poisson random measure $\Phi = \sum_{i=1}^N \delta_{\phi_i}$ with intensity measure μ is given and such that $\eta = \max(\Phi)$.

3.2.1 Extremal and sub-extremal point measures

Let $K \subset T$ be a closed subset of T. We introduce here the notion of K-extremal points that will play a key role in this work. We use the following notations : if f_1, f_2 are two functions defined (at least) on K, we write

$$\begin{aligned} f_1 =_K f_2 \quad &\text{if and only if} \quad \forall t \in K,\ f_1(t) = f_2(t), \\ f_1 <_K f_2 \quad &\text{if and only if} \quad \forall t \in K,\ f_1(t) < f_2(t), \\ f_1 \not<_K f_2 \quad &\text{if and only if} \quad \exists t \in K,\ f_1(t) \geq f_2(t). \end{aligned}$$

Let $M \in M_p(\mathcal{C}_0)$. An atom $f \in [M]$ is called K-sub-extremal if $f <_K \max(M)$ and K-extremal otherwise. In words, a sub-extremal atom has no contribution to the maximum $\max(M)$ on K.

Definition 3.2.1. *Define the K-extremal random point measure Φ_K^+ and the K-sub-extremal random point measure Φ_K^- by*

$$\Phi_K^+ = \sum_{i=1}^N 1_{\{\phi_i \not<_K \eta\}} \delta_{\phi_i} \quad \textit{and} \quad \Phi_K^- = \sum_{i=1}^N 1_{\{\phi_i <_K \eta\}} \delta_{\phi_i}.$$

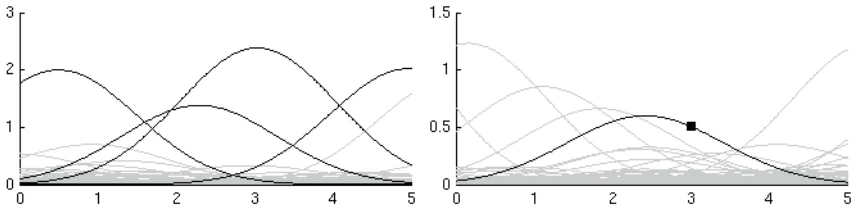

FIGURE 3.2 – Decomposition of the Poisson point measure Φ into the K-extremal point measure Φ_K^+ (black) and the K-sub-extremal point measure Φ_K^- (grey). Left : $K = [0, 5]$. Right : $K = \{3\}$ represented by a black square.

Figure 3.2 provides an illustration of the definition. It should be noted that Φ_K^+ and Φ_K^- are well defined measurable random point measures (see Lemma 3.A.3 in Appendix 3.A.3). Furthermore, it is straightforward from the definition that

$$\Phi = \Phi_K^+ + \Phi_K^-, \quad \max(\Phi_K^+) =_K \eta \quad \text{and} \quad \max(\Phi_K^-) <_K \eta.$$

Define the following measurable subsets of $M_p(\mathcal{C}_0)$ (see Lemma 3.A.4 in Appendix 3.A.3) :

$$C_K^+ = \Big\{ M \in M_p(\mathcal{C}_0);\ \forall f \in [M],\ f \not<_K \max(M) \Big\}, \tag{3.5}$$

$$C_K^-(g) = \Big\{ M \in M_p(\mathcal{C}_0);\ \forall f \in [M],\ f <_K g \Big\}, \tag{3.6}$$

where g is any continuous function defined (at least) on K. Clearly, it always holds

$$\Phi_K^+ \in C_K^+ \quad \text{and} \quad \Phi_K^- \in C_K^-(\eta).$$

The following theorem characterizes the joint distribution of (Φ_K^+, Φ_K^-) given that Φ_K^+ is finite. We note δ_0 the Dirac mass at 0.

Theorem 3.2.2. *For all measurable $A, B \subset M_p(\mathcal{C}_0)$,*

$$\mathbb{P}\big[(\Phi_K^+, \Phi_K^-) \in A \times B,\ \Phi_K^+(\mathcal{C}_0) = 0\big] = \exp[-\mu(\mathcal{C}_0)]\, \delta_0(A)\, \delta_0(B),$$

and, for $k \geq 1$,

$$\begin{aligned} &\mathbb{P}\big[(\Phi_K^+, \Phi_K^-) \in A \times B,\ \Phi_K^+(\mathcal{C}_0) = k\big] \\ = \ &\frac{1}{k!} \int_{\mathcal{C}_0^k} 1_{\{\sum_{i=1}^k \delta_{f_i} \in A \cap C_K^+\}} \mathbb{P}\left[\Phi \in B \cap C_K^- \left(\bigvee\nolimits_{i=1}^k f_i\right)\right] \mu^{\otimes k}(df_1, \ldots, df_k). \end{aligned}$$

Theorem 3.2.2 fully characterizes the joint distribution of (Φ_K^+, Φ_K^-) provided that $\Phi_K^+(\mathcal{C}_0)$ is almost surely finite. We now focus on this last condition.

Proposition 3.2.3.
The K-extremal point measure Φ_K^+ is a.s. finite if and only if one of the following condition holds :
(i) $\mu(\mathcal{C}_0) < +\infty$ *;*
(ii) $\mu(\mathcal{C}_0) = +\infty$ and $\inf_{t \in K} \eta(t) > 0$ almost surely.

It should be noted that any simple max-stable random field (with unit Fréchet margins) satisfies condition (ii) above. See for example Corollary 3.4 in [44].

Remark 3.2.4. *Using Theorem 3.2.2, it is easy to show that distribution of (Φ_K^+, Φ_K^-) has the following structure. Define the tail functional $\bar{\mu}_K$ by*

$$\bar{\mu}_K(g) = \mu(\{f \in \mathcal{C}_0;\ f \not<_K g\})$$

for any continuous function g defined (at least) on K. Suppose that Φ_K^+ is finite almost surely. Its distribution is then given by the so-called Janossy measures (see e.g. Daley and Vere-Jones [20] section 5.3). The Janossy measure of order k of the K-extremal point measure Φ_K^+ is given by

$$J_k(df_1, \ldots, df_k) = \exp\left[-\bar{\mu}_K\left(\bigvee\nolimits_{i=1}^k f_i\right)\right] 1_{\{\sum_{i=1}^k \delta_{f_i} \in C_K^+\}} \mu^{\otimes k}(df_1, \ldots, df_k).$$

Furthermore, given that $\Phi_K^+ = \sum_{i=1}^k \delta_{f_i}$, the conditional distribution of Φ_K^- is equal to the distribution of a Poisson random measure with intensity $1_{\{f <_K \bigvee_{i=1}^k f_i\}} \mu(df)$. These results are not used in the sequel and we omit their proof for the sake of brevity.

3.2.2 Extremal functions

Let $t \in T$. We denote by μ_t the measure on $(0, +\infty)$ defined by

$$\mu_t(A) = \mu(\{f \in \mathcal{C}_0; f(t) \in A\}), \quad A \subset (0, +\infty) \text{ Borel set},$$

and by $\bar{\mu}_t$ the associated tail function defined by

$$\bar{\mu}_t(x) = \mu_t([x, +\infty)), \quad x > 0.$$

Note that

$$\mathbb{P}(\eta(t) < x) = \exp(-\mu(\{f \in \mathcal{C}_0; f(t) \geq x\})) = \exp(-\bar{\mu}_t(x)), \quad x > 0. \tag{3.7}$$

The following proposition states that, under a natural condition, there is almost surely a unique $\{t\}$-extremal point in Φ. This extremal point will be referred to as the *t-extremal function* and noted ϕ_t^+.

Proposition 3.2.5. *For $t \in T$, the following statements are equivalent :*

(i) $\Phi_{\{t\}}^+(\mathcal{C}_0) = 1$ *almost surely ;*

(ii) $\bar{\mu}_t(0^+) = +\infty$ *and* $\bar{\mu}_t$ *is continuous on* $(0, +\infty)$ *;*

(iii) *the distribution of* $\eta(t)$ *has no atom.*

If these conditions are met, we define the t-extremal function ϕ_t^+ by the relation $\Phi_{\{t\}}^+ = \delta_{\phi_t^+}$ a.s. For all measurable $A \subset \mathcal{C}_0$ we have

$$\mathbb{P}(\phi_t^+ \in A) = \int_A \exp[-\bar{\mu}_t(f(t))]\, \mu(df). \tag{3.8}$$

An important class of processes satisfying the conditions of Proposition 3.2.5 is the class of max-stable processes (see section 3.4.2 below).

3.2.3 Hitting scenarios

Proposition 3.2.5 gives the distribution of Φ_K^+ when $K = \{t\}$ is reduced to a single point. Going a step further, we consider the case when K is finite. In the sequel, we suppose that the following assumption is satisfied :

(A) $K = \{t_1, \ldots, t_k\}$ is finite and, for all $t \in K$, $\bar{\mu}_t$ is continuous and $\bar{\mu}_t(0^+) = +\infty$.

Roughly speaking, this ensures that the maximum $\eta(t) = \max(\Phi)(t)$ is uniquely reached for all $t \in K$. This will provide combinatorial simplifications. More precisely, under Assumption $(\mathbf{A})$, the event

$$\Omega_K = \bigcap_{t \in K} \{\Phi_{\{t\}}^+(\mathcal{C}_0) = 1\}$$

is of probability 1 and the extremal functions $\phi_{t_1}^+, \ldots, \phi_{t_k}^+$ are well defined. In the next definition, we introduce the notion of hitting scenario that reflects the way how these extremal functions hit the maximum η on K.

Let $\mathcal{P}_K$ be the set of partitions of K. It is convenient to think about K as an ordered set, say $t_1 < \cdots < t_k$. Then each partition τ can be written uniquely in the standardized form $\tau = (\tau_1, \ldots, \tau_\ell)$ where $\ell = \ell(\tau)$ is the length of the partition, $\tau_1 \subset K$ is the component of t_1, $\tau_2 \subset K$ is the component containing $\min(K \setminus \tau_1)$ and so on. With this convention, the components $\tau_1, \ldots, \tau_\ell$ of the partition are labeled so that

$$\min \tau_1 < \cdots < \min \tau_\ell.$$

Definition 3.2.6. *Suppose that Assumption* $(\mathbf{A})$ *is met. Define* $\sim$ *the (random) equivalence relation on* $K=\{t_1,\dots,t_k\}$ *by* $t\sim t'$ *if and only if* $\phi_t^+=\phi_{t'}^+$. *The partition* $\Theta=(\theta_1,\dots,\theta_{\ell(\Theta)})$ *of* K *into equivalence classes is called the hitting scenario. For* $j\in[\![1,\ell(\Theta)]\!]$, *let* φ_j^+ *be the extremal function associated to the component* θ_j, *i.e., such that* $\varphi_j^+=\phi_t^+$ *for all* $t\in\theta_j$.

We illustrate the definition with two examples in Figure 3.3.

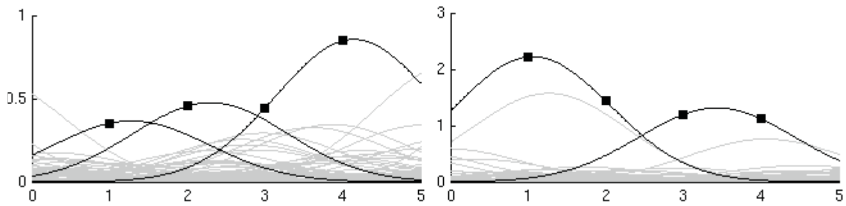

FIGURE 3.3 – Two realisations of the Poisson point measure Φ and of the corresponding hitting scenario Θ and extremal functions $\varphi_1^+,\dots,\varphi_{\ell(\Theta)}^+$ with $K=\{t_1,t_2,t_3,t_4\}$ represented by the black squares. Left : the hitting scenario is $\Theta=(\{t_1\},\{t_2\},\{t_3,t_4\})$, the extremal functions are $\varphi_1^+=\phi_{t_1}^+$, $\varphi_2^+=\phi_{t_2}^+$ and $\varphi_3^+=\phi_{t_3}^+=\phi_{t_4}^+$. Right : the hitting scenario is $\Theta=(\{t_1,t_2\},\{t_3,t_4\})$, the extremal functions (black) are $\varphi_1^+=\phi_{t_1}^+=\phi_{t_2}^+$ and $\varphi_2^+=\phi_{t_3}^+=\phi_{t_4}^+$.

Clearly a point $\phi\in[\Phi]$ is K-extremal if and only if it is t-extremal for some $t\in K$, so that $[\Phi_K^+]=\{\phi_t^+,\ t\in K\}$. Furthermore, the random measure Φ_K^+ is almost surely simple, i.e. any atoms have a simple multiplicity, otherwise the condition $\Phi_{\{t\}}^+(\mathcal{C}_0)=1$ a.s. would not be satisfied for some $t\in K$. These considerations entail that

$$\Phi_K^+=\sum_{j=1}^{\ell(\Theta)}\delta_{\varphi_j^+}. \tag{3.9}$$

In particular, the length $\ell(\Theta)$ of the hitting scenario is equal to $\Phi_K^+(\mathcal{C}_0)$. Furthermore the extremal functions satisfy

$$\forall j\in[\![1,\ell]\!],\ \forall t\in\theta_j,\quad \varphi_j^+(t)>\bigvee_{j'\neq j}\varphi_{j'}^+(t). \tag{3.10}$$

The distribution of the hitting scenario and extremal functions is given by the following proposition. The proof relies on Theorem 3.2.2.

Proposition 3.2.7. *Suppose Assumption* $(\mathbf{A})$ *is met.*
Then, for any partition $\tau=(\tau_1,\dots,\tau_\ell)\in\mathcal{P}_K$, *and any Borel sets* $A\subset\mathcal{C}_0^\ell$, $B\subset M_p(\mathcal{C}_0)$, *we have*

$$\begin{aligned}&\mathbb{P}[\Theta=\tau,\ (\varphi_1^+,\dots,\varphi_\ell^+)\in A,\ \Phi_K^-\in B] \qquad (3.11)\\ =\ &\int_A 1_{\{\forall j\in[\![1,\ell]\!],\ f_j>_{\tau_j}\bigvee_{j'\neq j}f_{j'}\}}\mathbb{P}\left[\Phi\in B\cap C_K^-\left(\bigvee\nolimits_{j=1}^{\ell}f_j\right)\right]\mu^{\otimes\ell}(df_1,\dots,df_\ell).\end{aligned}$$

3.3 Regular conditional distribution of max-id processes

We now focus on conditional distributions. We will need some notations.

If $\mathbf{s} = (s_1, \ldots, s_l) \in T^l$ and $f \in \mathcal{C}_0$, we note $f(\mathbf{s}) = (f(s_1), \ldots, f(s_l))$. Let $\mu_{\mathbf{s}}$ be the exponent measure of the max-i.d. random vector $\eta(\mathbf{s})$, i.e. the measure on $[0, +\infty)^l \setminus \{0\}$ defined by

$$\mu_{\mathbf{s}}(A) = \mu(\{f \in \mathcal{C}_0;\ f(\mathbf{s}) \in A\}), \quad A \subset [0, +\infty)^l \setminus \{0\} \text{ Borel set.}$$

Define the corresponding tail function

$$\bar{\mu}_{\mathbf{s}}(\mathbf{x}) = \mu(\{f \in \mathcal{C}_0;\ f(\mathbf{s}) \not< \mathbf{x} \text{ and } f(\mathbf{s}) \neq 0\}), \quad \mathbf{x} \in [0, +\infty)^l.$$

Let $\{P_{\mathbf{s}}(\mathbf{x}, df);\ \mathbf{x} \in [0, +\infty)^l \setminus \{0\}\}$ be a regular version of the conditional measure $\mu(df)$ given $f(\mathbf{s}) = \mathbf{x}$ (see Lemma 3.A.2 in Appendix 3.A.2). Then for any measurable function

$$F : [0, +\infty)^l \times \mathcal{C}_0 \to \ [0, +\infty)$$

vanishing on $\{0\} \times \mathcal{C}_0$, we have

$$\int_{\mathcal{C}_0} F(f(\mathbf{s}), f)\mu(df) \quad = \quad = \int_{[0,+\infty)^l \setminus \{0\}} \int_{\mathcal{C}_0} F(\mathbf{x}, f) P_{\mathbf{s}}(\mathbf{x}, df)\mu_{\mathbf{s}}(d\mathbf{x}). \tag{3.12}$$

Let $\mathbf{t} = (t_1, \ldots, t_k)$ and $\mathbf{y} = (y_1, \ldots, y_k) \in [0, +\infty)^k$. Before considering the conditional distribution of η with respect to $\eta(\mathbf{t}) = \mathbf{y}$, we give in the next theorem an explicit expression of the distribution of $\eta(\mathbf{t})$. We note $K = \{t_1, \ldots, t_k\}$. For any non empty $L \subset K$, we define $\tilde{L} = \{i \in [\![1, k]\!] : t_i \in L\}$ and set $\mathbf{t}_L = (t_i)_{i \in \tilde{L}}$, $\mathbf{y}_L = (y_i)_{i \in \tilde{L}}$ and $L^c = K \setminus L$.

Theorem 3.3.1. *Suppose assumption* (**A**) *is satisfied. For $\tau \in \mathcal{P}_K$, define the measure $\nu_{\mathbf{t}}^{\tau}$ on $[0, +\infty)^k$ by*

$$\nu_{\mathbf{t}}^{\tau}(C) = \mathbb{P}(\eta(\mathbf{t}) \in C;\ \Theta = \tau), \quad C \subset [0, +\infty)^k \text{ Borel set.}$$

Then,

$$\nu_{\mathbf{t}}^{\tau}(d\mathbf{y}) \quad = \quad \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})] \bigotimes_{j=1}^{\ell} \Big\{ P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\})\ \mu_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j}) \Big\} \tag{3.13}$$

and the distribution $\nu_{\mathbf{t}}$ of $\eta(\mathbf{t})$ is equal to $\nu_{\mathbf{t}} = \sum_{\tau \in \mathcal{P}_K} \nu_{\mathbf{t}}^{\tau}$.

Under some extra regularity assumptions, one can even get an explicit density function for $\nu_{\mathbf{t}}$ (see the section 3.4.3 on regular models below).

We are now ready to state our main result. In Theorem 3.3.2 below, we consider the regular conditional distribution of the point process Φ with respect to $\eta(\mathbf{t}) = \mathbf{y}$. Then, thanks to the relation $\eta = \max(\Phi)$, we deduce easily in Corollary 3.3.3 below the regular conditional distribution of η with respect to $\eta(\mathbf{t}) = \mathbf{y}$.

Recall that the point process has been decomposed into two parts : a hitting scenario Θ together with extremal functions $(\varphi_1^+,\dots,\varphi_{\ell(\Theta)}^+)$ and a K-sub-extremal point process Φ_K^-. Taking this decomposition into account, we introduce the following regular conditional distributions :

$$\begin{aligned}
\pi_{\mathbf{t}}(\mathbf{y},\,\cdot\,) &= \mathbb{P}[\Theta\in\cdot\ \mid \eta(\mathbf{t})=\mathbf{y}]\\
Q_{\mathbf{t}}(\mathbf{y},\tau,\,\cdot\,) &= \mathbb{P}[(\varphi_j^+)\in\cdot\mid \eta(\mathbf{t})=\mathbf{y},\Theta=\tau]\\
R_{\mathbf{t}}(\mathbf{y},\tau,(f_j),\,\cdot\,) &= \mathbb{P}[\Phi_K^-\in\cdot\mid \eta(\mathbf{t})=\mathbf{y},\Theta=\tau,(\varphi_j^+)=(f_j)].
\end{aligned}$$

We use here the short notations $\ell=\ell(\tau)$, $(\varphi_j^+)=(\varphi_1^+,\dots,\varphi_\ell^+)$ and similarly $(f_j)=(f_1,\dots,f_\ell)$. The following theorem provides explicit expressions for these regular conditional distributions.

Theorem 3.3.2. *Suppose assumption* **(A)** *is satisfied.*

1. *For any $\tau\in\mathcal{P}_K$, it holds $\nu_{\mathbf{t}}(d\mathbf{y})$-a.e.*

$$\pi_{\mathbf{t}}(\mathbf{y},\tau)=\frac{d\nu_{\mathbf{t}}^\tau}{d\nu_{\mathbf{t}}}(\mathbf{y}) \tag{3.14}$$

where $\nu_{\mathbf{t}}$ and $\nu_{\mathbf{t}}^\tau$ are defined in Theorem 3.3.1 and $d\nu_{\mathbf{t}}^\tau/d\nu_{\mathbf{t}}$ denotes the Radon-Nykodym derivative of $\nu_{\mathbf{t}}^\tau$ w.r.t. $\nu_{\mathbf{t}}$.

2. *It holds $\nu_{\mathbf{t}}(d\mathbf{y})\pi_{\mathbf{t}}(\mathbf{y},d\tau)$-a.e.*

$$Q_{\mathbf{t}}(\mathbf{y},\tau,df_1\cdots df_\ell)=\bigotimes_{j=1}^{\ell}\left\{\frac{1_{\{f_j(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\}}P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},df_j)}{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},\{f(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\})}\right\}. \tag{3.15}$$

In words, conditionally on $\eta(\mathbf{t})=\mathbf{y}$ and $\Theta=\tau$, the extremal functions $(\varphi_1^+,\dots,\varphi_\ell^+)$ are independent and φ_j^+ follows the distribution $P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},df)$ conditioned to the constraint $f(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}$.

3. *Let $C_{\mathbf{t}}^-(\mathbf{y})=\{M\in M_p(\mathcal{C}_0);\ \forall f\in[M],\ f(\mathbf{t})<\mathbf{y}\}$. It holds a.e.*

$$R_{\mathbf{t}}(\mathbf{y},\tau,(f_j),B)\equiv R_{\mathbf{t}}(\mathbf{y},B)=\frac{\mathbb{P}[\Phi\in B\cap C_{\mathbf{t}}^-(\mathbf{y})]}{\mathbb{P}[\Phi\in C_{\mathbf{t}}^-(\mathbf{y})]} \tag{3.16}$$

for any measurable $B\in M_p(\mathcal{C}_0)$. In words, conditionally on $\eta(\mathbf{t})=\mathbf{y}$, Φ_K^- is independent of Θ and $(\varphi_1^+,\dots,\varphi_{\ell(\Theta)}^+)$ and has the same distribution as a Poisson point measure with intensity $1_{\{f(\mathbf{t})<\mathbf{y}\}}\mu(df)$.

As a consequence, we deduce the regular conditional distribution of η with respect to $\eta(\mathbf{t})=\mathbf{y}$.

Theorem 3.3.3. *It holds $\nu_{\mathbf{t}}(d\mathbf{y})$-a.e.*

$$\begin{aligned}
&\mathbb{P}[\eta(\mathbf{s})<\mathbf{z}\mid\eta(\mathbf{t})=\mathbf{y}]\\
=\;&\exp[-\mu(\{f(\mathbf{s})\not<\mathbf{z},\ f(\mathbf{t})<\mathbf{y}\})]\sum_{\tau\in\mathcal{P}_K}\pi_{\mathbf{t}}(\mathbf{y},\tau)\prod_{j=1}^{\ell}\frac{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},\{f(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c},\ \ f(\mathbf{s})<\mathbf{z}\})}{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j},\{f(\mathbf{t}_{\tau_j^c})<\mathbf{y}_{\tau_j^c}\})}
\end{aligned}$$

for any $l\geq 1$, $\mathbf{s}\in T^l$ and $\mathbf{z}\in[0,+\infty)^l$.

Remark 3.3.4. Let us mention that Theorem 3.3.2 suggests a three-step procedure for sampling from the conditional distribution of η given $\eta(\mathbf{t}) = \mathbf{y}$:

1. Draw a random partition τ with distribution $\pi_{\mathbf{t}}(\mathbf{y}, \cdot)$.
2. Given $\tau = \{\tau_1, \dots, \tau_\ell\}$, draw ℓ independent functions $\psi_1, \dots, \psi_\ell$, with ψ_j following the distribution $P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df)$ conditioned on $f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}$.
3. Independently of the above two steps, draw $\sum_{i\in I} \delta_{\phi_i}$ a Poisson point measure on $\mathcal{C}_0$ with intensity $1_{\{f(\mathbf{t})<\mathbf{y}\}}\mu(df)$. It can be obtained from a Poisson point measure with intensity $\mu(df)$ by removing those points not satisfying the constraint $f(\mathbf{t}) < \mathbf{y}$.

Then, the random field

$$\tilde{\eta}(t) = \max\{\psi_1(t), \dots, \psi_\ell(t)\} \vee \max\{\phi_i(t),\ i \in I\}, \quad t \in T,$$

has the required conditional distribution.

The issues and computational aspects of conditional sampling are addressed in the paper [36]. The special case of Brown-Resnick max-stable processes is considered and tractable expressions are derived and the above three-step procedure is implemented effectively.

3.4 Examples

As an illustration, we apply in this section our general results to specific cases.

3.4.1 The case of a single conditioning point

It is worth noting that the case of a single conditioning point, i.e. $k = 1$, gives rise to major simplifications. There exists indeed a unique partition of the set $K = \{t\}$ so that the notion of hitting scenario is irrelevant. Furthermore, there is a.s. a single K-extremal function φ_1^+ which is equal to the t-extremal function ϕ_t^+. In this case, Theorems 3.3.2 and 3.3.3 simplify into the following proposition.

Proposition 3.4.1. *Let $t \in T$ and suppose that conditions (i)-(iii) in Proposition 3.2.5 are met. Then, conditionally on $\eta(t) = y$, ϕ_t^+ and Φ_K^- are independent ; the conditional distribution of ϕ_t^+ is equal to $P_t(y, \cdot)$; the conditional distribution of Φ_K^- is equal to the distribution of a Poisson point measure with intensity $1_{\{f(t)<y\}}\mu(df)$. Furthermore, for $l \geq 1$, $\mathbf{s} \in T^l$ and $\mathbf{z} \in [0, +\infty)^l$,*

$$\mathbb{P}[\eta(\mathbf{s}) < \mathbf{z} \mid \eta(t) = y] \quad = \quad P_t(y, \{f(\mathbf{s}) < \mathbf{z}\}) \exp[-\mu(\{f(\mathbf{s}) \not< \mathbf{z},\ f(t) < y\})]. \quad (3.17)$$

3.4.2 Max-stable models

We put the emphasis here on max-stable random fields. For convenience and without loss of generality, we focus on simple max-stable random fields η, i.e., with standard unit Fréchet margins

$$\mathbb{P}(\eta(t) \leq x) = \exp[-x^{-1}]1_{\{x>0\}}, \quad x \in \mathbb{R},\ t \in T.$$

A random field η is said to be simple max-stable if for any $n \geq 1$,

$$\eta \stackrel{\mathcal{L}}{=} n^{-1} \bigvee_{i=1}^{n} \eta_i$$

where $\{\eta_i,\ i \geq 1\}$ are i.i.d. copies of η. Any general max-stable random field can be related to such a simple max-stable random field η by simple transformation of the margins, see e.g. Corollary 3.6 in [44]. Furthermore, Corollary 4.5.6 in [31] states that η can be represented as

$$\{\eta(t)\}_{t\in T} \stackrel{\mathcal{L}}{=} \Big\{ \bigvee_{i\geq 1} \Gamma_i Y_i(t) \Big\}_{t\in T} \tag{3.18}$$

where $(\Gamma_i)_{i\geq 1}$ is the nonincreasing enumeration of the points of a Poisson point process on $(0,\infty)$ with intensity $x^{-2}dx$, $(Y_i)_{i\geq 1}$ is an i.i.d. sequence of continuous random processes on T, independent of $(\Gamma_i)_{i\geq 1}$ and such that

$$\mathbb{E}[Y_1(t)] = 1,\ t \in T, \quad \text{and} \quad \mathbb{E}[\|Y_1\|] < \infty.$$

Since a continuous simple max-stable random field is max-i.d., it has a Poisson point measure representation (3.1). The normalization to unit Fréchet margins entails that the vertex function is equal to 0 and that the exponent measure μ satifies, for all $t \in T$,

$$\mu_t(dy) = y^{-2} 1_{\{y>0\}} dy \quad \text{and} \quad \bar{\mu}_t(y) = y^{-1}, \quad y > 0.$$

The correspondence between the two representations (3.1) and (3.18) is the following : the point measure $\Phi = \sum_{i\geq 1} \delta_{\Gamma_i Y_i}$ on $\mathcal{C}_0$ is a Poisson point measure with intensity

$$\mu(A) = \int_0^\infty \mathbb{E}[1_{\{rY_1 \in A\}}]\, r^{-2} dr, \quad A \subset \mathcal{C}_0 \text{ Borel set},$$

The distribution of the Y_i's, denoted by σ, is called the spectral measure and is related to the exponent measure μ by the relation

$$\mu(A) = \int_0^\infty \int_{\mathcal{C}_0} 1_{\{rf\in A\}}\, \sigma(df) r^{-2} dr, \quad A \subset \mathcal{C}_0 \text{ Borel set}.$$

Taking into account this particular form of the exponent measure, we can relate the kernel $P_t(y,\cdot)$ to the spectral measure σ. For $x \in \mathbb{R}$, we note $(x)^+ = \max(x,0)$.

Proposition 3.4.2. *Let η be a continuous simple max-stable random field with spectral measure σ and $t \in T$. The $\{t\}$-extremal function ϕ_t^+ has conditional distribution*

$$\mathbb{P}[\phi_t^+ \in \cdot \mid \eta(t) = y] = P_t(y,\cdot) = \int_{\mathcal{C}_0} 1_{\{\frac{y}{f(t)} f \in \cdot\}} f(t) \sigma(df).$$

Furthermore, for $l \geq 1$, $\mathbf{s} \in T^l$ and $\mathbf{z} \in [0,+\infty)^l$,

$$\begin{aligned} &\mathbb{P}[\eta(\mathbf{s}) < \mathbf{z} \mid \eta(t) = y] \\ =\ & \exp\Big[-\int_{\mathcal{C}_0} \Big(\bigvee_{i=1}^{l} \frac{f(s_i)}{z_i} - \frac{f(t)}{y} \Big)^+ \sigma(df) \Big] \int_{\mathcal{C}_0} 1_{\{\bigvee_{i=1}^{l} \frac{f(s_i)}{z_i} < \frac{f(t)}{y}\}} f(t) \sigma(df). \end{aligned} \tag{3.19}$$

Equation (3.19) extends Lemma 3.4 in Weintraub [80] where only the bivariate case $l = 1$ is considered. Note the author considers min-stability rather than max-stability ; the correspondence is straightforward since, if η is simple max-stable, then η^{-1} is min-stable with exponential margins.

3.4.3 Regular models

We have considered so far the case of a single conditioning point which allows for major simplifications. In the general case, there are several conditioning points and the hitting scenario is non trivial. This introduces more complexity since the conditional distribution is expressed as a mixture over any possible hitting scenarios and involves an abstract Radon-Nykodym derivative. The framework of regular models can be helpful to get more tractable formulas.

The exponent measure μ is said to be *regular* (with respect to the Lebesgue measure) if for any $l \geq 1$ and $\mathbf{s} \in T^l$ with pairwise distinct components, the measure $\mu_{\mathbf{s}}(d\mathbf{z})$ is absolutely continuous with respect to the Lebesgue measure $d\mathbf{z}$ on $[0, +\infty)^l$. We denote by $\lambda_{\mathbf{s}}$ the corresponding Radon-Nykodym derivative, i.e., $\mu_{\mathbf{s}}(d\mathbf{z}) = \lambda_{\mathbf{s}}(\mathbf{z})d\mathbf{z}$.

Under this assumption, we can reformulate Theorems 3.3.1 and 3.3.2. For example, Equation (3.13) implies that the distribution $\nu_{\mathbf{t}}$ of $\eta(\mathbf{t})$ is absolutely continuous with respect to the Lebesgue measure with density

$$\frac{d\nu_{\mathbf{t}}}{d\mathbf{y}}(\mathbf{y}) = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})] \sum_{\tau \in \mathcal{P}_K} \prod_{j=1}^{\ell(\tau)} \int_{\{\mathbf{z}_j < \mathbf{y}_{\tau_j^c}\}} \lambda_{(\mathbf{t}_{\tau_j}, \mathbf{t}_{\tau_j^c})}(\mathbf{y}_{\tau_j}, \mathbf{z}_j) d\mathbf{z}_j.$$

Equation (3.14) giving the conditional distribution of the hitting scenario becomes

$$\pi_{\mathbf{t}}(\mathbf{y}, \tau) = \frac{\prod_{j=1}^{\ell(\tau)} \int_{\{\mathbf{z}_j < \mathbf{y}_{\tau_j^c}\}} \lambda_{(\mathbf{t}_{\tau_j}, \mathbf{t}_{\tau_j^c})}(\mathbf{y}_{\tau_j}, \mathbf{z}_j) d\mathbf{z}_j}{\sum_{\tau' \in \mathcal{P}_K} \prod_{j=1}^{\ell(\tau')} \int_{\{\mathbf{z}_j < \mathbf{y}_{\tau_j'^c}\}} \lambda_{(\mathbf{t}_{\tau_j'}, \mathbf{t}_{\tau_j'^c})}(\mathbf{y}_{\tau_j'}, \mathbf{z}_j) d\mathbf{z}_j}.$$

The conditional distribution of the extremal functions $Q_{\mathbf{t}}(\mathbf{y}, \tau, \cdot)$ in Equation (3.15) is based on the kernel $P_{\mathbf{t}}(\mathbf{y}, df)$. Using the existence of a Radon-Nykodym derivative for the finite dimensional margins of μ, we obtain

$$P_{\mathbf{t}}(\mathbf{y}, f(\mathbf{s}) \in d\mathbf{z}) = \frac{\lambda_{(\mathbf{t},\mathbf{s})}(\mathbf{y}, \mathbf{z})}{\lambda_{\mathbf{t}}(\mathbf{y})} d\mathbf{z}.$$

This approach is exploited in [36] for Brown-Resnick max-stable processes. Indeed, the model turns out to be regular.

3.5 Proofs

3.5.1 Proof of Theorem 3.2.2 and Proposition 3.2.3

For the proof of Theorem 3.2.2, we need the following lemma giving a useful characterization of the K-extremal random point measure. If $M_1, M_2 \in M_p(\mathcal{C}_0)$ are such that $M_2 - M_1 \in M_p(\mathcal{C}_0)$, we call M_1 a sub-point measure of M_2.

Lemma 3.5.1. *The K-extremal point measure Φ_K^+ is the unique sub-point measure $\tilde{\Phi}$ of Φ such that*

$$\tilde{\Phi} \in C_K^+ \quad \text{and} \quad \Phi - \tilde{\Phi} \in C_K^-(\max(\tilde{\Phi})).$$

Proof of Lemma 3.5.1:
First the condition $\Phi - \tilde{\Phi} \in C_K^-(\max(\tilde{\Phi}))$ implies

$$\max(\Phi - \tilde{\Phi}) <_K \max(\tilde{\Phi}) \quad \text{and} \quad \max(\tilde{\Phi}) =_K \max(\Phi).$$

Let $f \in [\Phi - \tilde{\Phi}]$. The condition $\Phi - \tilde{\Phi} \in C_K^-(\max(\tilde{\Phi}))$ implies $f <_K \max(\tilde{\Phi})$. Since $\tilde{\Phi}$ is a sub-point measure of Φ, $\max(\tilde{\Phi}) \leq \max(\Phi)$ so that $f <_K \max(\Phi)$ and f is K-sub-extremal in Φ.
Conversely for $f \in [\tilde{\Phi}]$, the condition $\tilde{\Phi} \in C_K^+$ implies the existence of $t_0 \in K$ such that $f(t_0) = \max(\tilde{\Phi})(t_0)$. Hence $f(t_0) = \max(\Phi)(t_0)$ and f is K-extremal in Φ.
□

Proof of Theorem 3.2.2:
First note that $\Phi_K^+(\mathcal{C}_0) = 0$ if and only if $\Phi = 0$. This occurs with probability $\exp[-\mu(\mathcal{C}_0)]$ and in this case $\Phi_K^+ = \Phi_K^- = 0$. The first claim follows.
Next, let $k \geq 1$. According to Lemma 3.5.1, $\Phi_K^+(\mathcal{C}_0) = k$ if and only if there exists a k-uplet $(\phi_1, \ldots, \phi_k) \in [\Phi]^k$ such that

$$\sum_{i=1}^k \delta_{\phi_i} \in C_K^+ \quad \text{and} \quad \Phi - \sum_{i=1}^k \delta_{\phi_i} \in C_K^- \left(\bigvee\nolimits_{i=1}^k \phi_i \right).$$

When this holds, the k-uplet $(\phi_1, \ldots, \phi_k)$ is unique up to a permutation of the coordinates and we have

$$\sum_{i=1}^k \delta_{\phi_i} = \Phi_K^+ \quad \text{and} \quad \Phi - \sum_{i=1}^k \delta_{\phi_i} = \Phi_K^-.$$

Hence the sum

$$\int_{\mathcal{C}_0^k} 1_{\left\{\sum_{i=1}^k \delta_{\phi_i} \in A \cap C_K^+,\ \Phi - \sum_{i=1}^k \delta_{\phi_i} \in C_K^-\left(\bigvee_{i=1}^k \phi_i\right)\right\}} \Phi(d\phi_1) \cdots \big(\Phi - \sum_{j=1}^{k-1} \delta_{\phi_j}\big)(d\phi_k)$$

is equal to $k! 1_{\{(\Phi_K^+, \Phi_K^-) \in A \times B\}}$ if $\Phi_K^+(\mathcal{C}_0) = k$ and 0 otherwise. Using this and Slyvniak's formula (see Appendix 3.A.1), we get

$$\begin{aligned}
& k! \mathbb{P}\big[(\Phi_K^+, \Phi_K^-) \in A \times B,\ \Phi_K^+(\mathcal{C}_0) = k\big] \\
= \ & \mathbb{E}\Big[\int_{\mathcal{C}_0^k} 1_{\left\{\sum_{i=1}^k \delta_{\phi_i} \in A \cap C_K^+,\ \Phi - \sum_{i=1}^k \delta_{\phi_i} \in B \cap C_K^-\left(\bigvee_{i=1}^k \phi_i\right)\right\}} \Phi(d\phi_1) \cdots \big(\Phi - \sum_{j=1}^{k-1} \delta_{\phi_j}\big)(d\phi_k)\Big] \\
= \ & \int_{\mathcal{C}_0^k} 1_{\{\sum_{i=1}^k \delta_{f_i} \in A \cap C_K^+\}} \mathbb{P}\left[\Phi \in B \cap C_K^- \left(\bigvee\nolimits_{i=1}^k f_i\right)\right] \mu^{\otimes k}(df_1, \ldots, df_k).
\end{aligned}$$

This proves Theorem 3.2.2. $\square$

Proof of Proposition 3.2.3:
In the case $\mu(\mathcal{C}_0) < +\infty$, Φ and *a fortiori* Φ_K^+ are a.s. finite. Suppose now $\mu(\mathcal{C}_0) = +\infty$, so that Φ is a.s. infinite. If $\inf_{t\in K} \eta(t) = 0$, then there is $t_0 \in K$ such that $\eta(t_0) = 0$ (recall η is continuous and K compact). This implies that $\phi(t_0) = 0$ for all $\phi \in [\Phi]$ and hence $\Phi_K^+ = \Phi$ is infinite. If $\inf_{t\in K} \eta(t) = \varepsilon > 0$, then the support of Φ_K^+ is included in the set $\{f \in \mathcal{C}_0;\ f(K) \geq \varepsilon\}$. From the definition of $M_p(\mathcal{C}_0)$, this set contains only a finite number of atoms of Φ so that Φ_K^+ must be finite. $\square$

3.5.2 Proof of Propositions 3.2.5 and 3.2.7

Proof of Proposition 3.2.5:
According to equation (3.7), for all $x > 0$,

$$\mathbb{P}[\eta(t) = x] = \exp[-\bar{\mu}_t(x^+)] - \exp[-\bar{\mu}_t(x)],$$

and $\mathbb{P}[\eta(t) = 0] = \exp[-\bar{\mu}_t(0^+)]$. The equivalence between (ii) and (iii) follows.
The equivalence between (i) and (ii) is a consequence of Theorem 3.2.2 with $K = \{t\}$, $k = 1$ and $A = B = M_p(\mathcal{C}_0)$: we get

$$\begin{aligned}
\mathbb{P}[\Phi_{\{t\}}^+(\mathcal{C}_0) = 1] &= \int_{\mathcal{C}_0} 1_{\{\delta_f \in C_{\{t\}}^+\}} \mathbb{P}[\Phi \in C_{\{t\}}^-(f)]\mu(df) \\
&= \int_{[0,+\infty)} \exp[-\bar{\mu}_t(y)]\, \mu_t(dy).
\end{aligned}$$

It remains to prove that this probability is equal to 1 if and only if (ii) is satisfied. To this aim, we compute

$$\begin{aligned}
\mathbb{P}[\Phi_{\{t\}}^+(\mathcal{C}_0) = 1] &= \int_{(0,+\infty)^2} e^x 1_{\{x \leq -\bar{\mu}_t(y)\}}\, dx \mu_t(dy) \\
&= \int_{(0,+\infty)} e^x \mu_t(A_x)\, dx,
\end{aligned} \tag{3.20}$$

where $A_x = \{y > 0 : \bar{\mu}_t(y) \leq x\}$. Since $-\bar{\mu}_t$ is càg-làd, decreasing and tends to ∞ at 0, $A_x = (\inf A_x, \infty) \neq \varnothing$ for all $x > 0$. Furthermore using equation (3.20) and the fact that $\mu_t(A_x) \leq x$, we get that $\mathbb{P}[\Phi_{\{t\}}^+(\mathcal{C}_0) = 1] = 1$ if and only if $\mu_t(A_x) = x$ for all $x > 0$. We see easily that this is equivalent to condition (ii) and this completes the equivalence between (i) and (ii).

We now prove Equation (3.8). Assuming that conditions (i)-(iii) are met, it holds

$$\mathbb{P}(\phi_t^+ \in A) = \mathbb{P}[\Phi_{\{t\}}^+ \in \tilde{A},\ \Phi_{\{t\}}^+(\mathcal{C}_0) = 1]$$

with $\tilde{A} = \{\delta_f,\ f \in A\}$. Theorem 3.2.2 with $K = \{t\}$, $k = 1$ and $B = M_p(\mathcal{C}_0)$ entails

$$\begin{aligned}\mathbb{P}(\phi_t^+ \in A) &= \int_{\mathcal{C}_0} 1_{\{\delta_f \in \tilde{A} \cap C^+_{\{t\}}\}} \mathbb{P}[\Phi \cap C^-_{\{t\}}(f)]\, \mu(df) \\ &= \int_{\mathcal{C}_0} 1_{\{f \in A\}} \exp[-\bar{\mu}_t(f(t))]\, \mu(df).\end{aligned}$$

This proves Equation (3.8). □

Proof of Proposition 3.2.7:
First note that the inequalities (3.10) characterize the hitting scenario. Let $\tau = (\tau_1, \ldots, \tau_\ell) \in \mathcal{P}_K$ and define the sets

$$\tilde{C}_\tau = \Big\{(f_1, \ldots, f_\ell) \in \mathcal{C}_0^\ell;\ \forall j \in [\![1, \ell]\!],\ f_j >_{\tau_j} \bigvee_{j' \neq j} f_{j'}\Big\}.$$

and

$$C_\tau = \Big\{\sum_{j=1}^{\ell} \delta_{f_j} \in M_p(\mathcal{C}_0);\ (f_1, \ldots, f_\ell) \in \tilde{C}_\tau\Big\}.$$

Note that $C_\tau \subset C_K^+$ and that $\Theta = \tau$ if and only if $\Phi_K^+ \in C_\tau$.
Furthermore, $\Theta = \tau$ and $(\varphi_1^+, \ldots, \varphi_\ell^+) \in A$ if and only if $\Phi_K^+ \in A_\tau$ with

$$A_\tau = \Big\{\sum_{j=1}^{\ell} \delta_{f_j} \in M_p(\mathcal{C}_0);\ (f_1, \ldots, f_\ell) \in C_\tau \cap A\Big\}.$$

Hence the following events are equal

$$\{\Theta = \tau,\ (\varphi_1^+, \ldots, \varphi_\ell^+) \in A,\ \Phi_K^- \in B\} = \{\Phi_K^+ \in A_\tau,\ \Phi_K^- \in B,\ \Phi_K^+(\mathcal{C}_0) = \ell\}$$

and Theorem 3.2.2 implies

$$\begin{aligned}&\mathbb{P}[\Theta = \tau,\ (\varphi_1^+, \ldots, \varphi_\ell^+) \in A,\ \Phi_K^- \in B] \\ = &\ \frac{1}{\ell!} \int_{\mathcal{C}_0^\ell} 1_{\{\sum_{j=1}^{\ell} \delta_{f_j} \in A_\tau\}} \mathbb{P}\left[\Phi \in B \cap C_K^-\left(\bigvee_{j=1}^{\ell} f_j\right)\right] \mu^{\otimes \ell}(df_1, \ldots, df_\ell). \qquad (3.21)\end{aligned}$$

Finally, $\sum_{j=1}^{\ell} \delta_{f_j} \in A_\tau$ if and only if there exists a permutation σ of $[\![1, \ell]\!]$ such that $(f_{\sigma(1)}, \ldots, f_{\sigma(\ell)}) \in A \cap \tilde{C}_\tau$. Such a permutation is unique and this proves the equivalence of Equations (3.11) and (3.21). □

3.5.3 Proofs of Theorems 3.3.1, 3.3.2 and 3.3.3

Proof of Theorems 3.3.1 and 3.3.2:
Note that $\eta(\mathbf{t})$ can be expressed in terms of the hitting scenario and the extremal function as follows. For $\tau \in \mathcal{P}_K$, define the mapping $\Gamma_\tau : \mathcal{C}_0^\ell \to [0, +\infty)^k$ by

$$\Gamma_\tau(f_1, \ldots, f_\ell) = (y_1, \ldots, y_k) \quad \text{with } y_i = f_j(t_i) \text{ if } t_i \in \tau_j.$$

Definition (3.2.6) entails that for all $t \in \theta_j$, $\eta(t) = \varphi_j^+(t)$. This can be rewritten as $\eta(\mathbf{t}) = \Gamma_\Theta(\phi_1^+, \ldots, \phi_\ell^+)$. Using this, the probability

$$P(\tau, A, B, C) = \mathbb{P}[\Theta = \tau,\ (\varphi_1^+, \ldots, \varphi_\ell^+) \in A,\ \Phi_K^- \in B,\ \eta(\mathbf{t}) \in C]$$

can be computed thanks to Proposition 3.2.7 :

$$\begin{aligned}
& P(\tau, A, B, C) \\
= \ & \mathbb{P}[\Theta = \tau,\ (\varphi_1^+, \ldots, \varphi_\ell^+) \in A \cap \Gamma_\tau^{-1}(C),\ \Phi_K^- \in B,\ \eta(\mathbf{t}) \in C] \\
= \ & \int_{A \cap \Gamma_\tau^{-1}(C)} 1_{\left\{\forall j \in [\![1,\ell]\!],\ f_j >_{\tau_j} \bigvee_{j' \neq j} f_{j'}\right\}} \mathbb{P}\left[\Phi \in B \cap C_K^-\left(\bigvee_{j=1}^{\ell} f_j\right)\right] \mu^{\otimes \ell}(df_1, \ldots, df_\ell).
\end{aligned}$$

Now for each $j \in [\![1,\ell]\!]$, we condition the measure $\mu(df_j)$ with respect to $f_j(\mathbf{t}_{\tau_j})$: Equation (3.12) entails

$$\begin{aligned}
& P(\tau, A, B, C) \qquad (3.22) \\
= \ & \int_C \int_A 1_{\left\{\forall j \in [\![1,\ell]\!],\ f_j >_{\tau_j} \bigvee_{j' \neq j} f_{j'}\right\}} \mathbb{P}\left[\Phi \in B \cap C_K^-\left(\bigvee_{j=1}^{\ell} f_j\right)\right] \otimes_{j=1}^{\ell} \left\{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)\, \mu_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j})\right\} \\
= \ & \int_C \int_A 1_{\{\forall j \in [\![1,\ell]\!],\ f_j(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\}} \times \mathbb{P}[\Phi \in B \cap C_{\mathbf{t}}^-(\mathbf{y})] \otimes_{j=1}^{\ell} \left\{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)\, \mu_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j})\right\}.
\end{aligned}$$

In the last equality, we use the fact that $f_j(\mathbf{t}_{\tau_j}) = \mathbf{y}_{\tau_j}$ a.s. under $P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)$, whence

$$\begin{aligned}
\left\{\forall j \in [\![1,\ell]\!],\ f_j >_{\tau_j} \bigvee_{j' \neq j} f_{j'}\right\} &= \left\{\forall j \in [\![1,\ell]\!],\ f_j <_{\tau_j^c} \bigvee_{j' \neq j} f_{j'}\right\} \\
&= \left\{\forall j \in [\![1,\ell]\!],\ f_j(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\right\}
\end{aligned}$$

and $C_K^-(\bigvee_{j=1}^{\ell} f_j) = C_{\mathbf{t}}^-(\mathbf{y})$.

We now prove Theorem 3.3.1. Setting $A = \mathcal{C}_0^\ell$ and $B = M_p(\mathcal{C}_0)$ in Equation (3.22), we obtain

$$\begin{aligned}
& \mathbb{P}[\Theta = \tau,\ \eta(\mathbf{t}) \in C] \\
= \ & \int_C \int_{\mathcal{C}_0^\ell} 1_{\{\forall j \in [\![1,\ell]\!],\ f_j(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\}} \mathbb{P}[\Phi \in C_{\mathbf{t}}^-(\mathbf{y})] \otimes_{j=1}^{\ell} \left\{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)\, \mu_{\mathbf{t}_{\tau_j}}(d\mathbf{y}_{\tau_j})\right\}.
\end{aligned}$$

Using the fact that $\mathbb{P}[\Phi \in C_{\mathbf{t}}^-(\mathbf{y})] = \exp[-\bar{\mu}_{\mathbf{t}}(\mathbf{y})]$ and performing integration with respect to $\otimes_{j=1}^{\ell} P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, df_j)$, we obtain Equation (3.13) and this proves Theorem 3.3.1.

We now consider Theorem 3.3.2. Combining Equations (3.13)-(3.16) together with Equation (3.22), we get

$$\begin{aligned}
& \mathbb{P}[\Theta = \tau,\ (\varphi_1^+, \ldots, \varphi_\ell^+) \in A,\ \Phi_K^- \in B,\ \eta(\mathbf{t}) \in C] \\
= \ & \int_C \int_A \frac{\mathbb{P}[\Phi \in B \cap C_{\mathbf{t}}^-(\mathbf{y})]}{\mathbb{P}[\Phi \in C_{\mathbf{t}}^-(\mathbf{y})]} Q_{\mathbf{t}}(\mathbf{y}, \tau, df_1 \cdots df_\ell) \nu_{\mathbf{t}}^{\tau}(d\mathbf{y}) \\
= \ & \int_C \int_A R_{\mathbf{t}}(\mathbf{y}, \tau, (f_j), B) Q_{\mathbf{t}}(\mathbf{y}, \tau, df_1 \cdots df_\ell) \pi_{\mathbf{t}}(\mathbf{y}, \tau) \nu_{\mathbf{t}}(d\mathbf{y}). \qquad (3.23)
\end{aligned}$$

In particular, with $A = \mathcal{C}_0^\ell$ and $B = M_p(\mathcal{C}_0)$, we obtain the relation

$$\mathbb{P}[\Theta = \tau,\ \eta(\mathbf{t}) \in C] = \int_C \pi_{\mathbf{t}}(\mathbf{y}, \tau)\nu_{\mathbf{t}}(d\mathbf{y})$$

characterizing the regular conditional distribution $\mathbb{P}[\Theta = \tau \mid \eta(\mathbf{t}) = \mathbf{y}]$ (see Appendix 3.A.2). This proves that Equation (3.14) provides the regular conditional distribution $\mathbb{P}[\Theta = \tau \mid \eta(\mathbf{t}) = \mathbf{y}]$. Similarly, Equation (3.23) entails that the regular conditional distributions $\mathbb{P}[(\varphi_j^+) \in \cdot \mid \eta(\mathbf{t}) = \mathbf{y}, \Theta = \tau]$ and $\mathbb{P}[\Phi_K^- \in \cdot \mid \eta(\mathbf{t}) = \mathbf{y}, \Theta = \tau, (\varphi_j^+) = (f_j)]$ are given respectively by $Q_{\mathbf{t}}(\mathbf{y}, \tau, \cdot)$ in Equation (3.15) and $R_{\mathbf{t}}(\mathbf{y}, \tau, (f_j), \cdot)$ in Equation (3.16).

We briefly comment on these formulas. The fact that the distribution $Q_{\mathbf{t}}(\mathbf{y}, \tau, \cdot)$ in Equation (3.15) factorizes into a tensorial product means that the extremal functions $\varphi_1^+, \ldots, \varphi_\ell^+$ are independent conditionally on $\eta(\mathbf{t}) = \mathbf{y}$ and $\Theta = \tau$. The fact that the distribution $R_{\mathbf{t}}(\mathbf{y}, \tau, (f_j), \cdot)$ in Equation (3.16) does not depend on τ and (f_j) means that conditionally on $\eta(\mathbf{t}) = \mathbf{y}$, Φ_K^- is independent of Θ and $(\varphi_1^+, \ldots, \varphi_{\ell(\Theta)}^+)$. The distribution $R_{\mathbf{t}}(\mathbf{y}, \cdot)$ can be seen as the distribution of the Poisson point measure Φ conditioned to lie in $C_{\mathbf{t}}^-(\mathbf{y})$, i.e., to have no atom in $\{f \in \mathcal{C}_0;\ f(\mathbf{t}) \not< \mathbf{y}\}$. It is equal to the distribution of a Poisson point measure with intensity $1_{\{f(\mathbf{t}) < \mathbf{y}\}}\mu(df)$. □

Proof of Theorem 3.3.3:
Remark that

$$\begin{aligned}
\{\eta(\mathbf{s}) < \mathbf{z}\} &= \{(\Phi_K^+, \Phi_K^-) \in C_{\mathbf{s}}^-(\mathbf{z}) \times C_{\mathbf{s}}^-(\mathbf{z})\} \\
&= \bigcup_{\tau \in \mathcal{P}_K} \{\Theta = \tau, \varphi_1^+(\mathbf{s}) < \mathbf{z}, \ldots, \varphi_\ell^+(\mathbf{s}) < \mathbf{z}, \Phi_K^- \in C_{\mathbf{s}}^-(\mathbf{z})\}
\end{aligned}$$

where $C_{\mathbf{s}}^-(\mathbf{z})$ is defined in Theorem 3.3.2. Using this, Theorem 3.3.2 entails

$$\begin{aligned}
&\mathbb{P}[\eta(\mathbf{s}) < \mathbf{z} \mid \eta(\mathbf{t}) = \mathbf{y}] \\
= &\sum_{\tau \in \mathcal{P}_K} \mathbb{P}[\Theta = \tau, \varphi_1^+(\mathbf{s}) < \mathbf{z}, \ldots, \varphi_\ell^+(\mathbf{s}) < \mathbf{z}, \Phi_K^- \in C_{\mathbf{s}}^-(\mathbf{z}) \mid \eta(\mathbf{t}) = \mathbf{y}] \\
= &\sum_{\tau \in \mathcal{P}_K} \pi_{\mathbf{t}}(\mathbf{y}, \tau) Q_{\mathbf{t}}(\mathbf{y}, \tau, \{f(\mathbf{s}) < \mathbf{z}\}^\ell) R_{\mathbf{t}}(\mathbf{y}, C_{\mathbf{s}}^-(\mathbf{z})).
\end{aligned}$$

The result follows since

$$Q_{\mathbf{t}}(\mathbf{y}, \tau, \{f(\mathbf{s}) < \mathbf{z}\}^\ell) = \prod_{j=1}^{\ell} \frac{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c},\ f(\mathbf{s}) < \mathbf{z}\})}{P_{\mathbf{t}_{\tau_j}}(\mathbf{y}_{\tau_j}, \{f(\mathbf{t}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\})}$$

and

$$\begin{aligned}
R_{\mathbf{t}}(\mathbf{y}, C_{\mathbf{s}}^-(bz)) &= \frac{\mathbb{P}[\Phi \in C_{\mathbf{s}}^-(\mathbf{z}) \cap C_{\mathbf{t}}^-(\mathbf{y})]}{\mathbb{P}[\Phi \in C_{\mathbf{t}}^-(\mathbf{y})]} \\
&= \frac{\exp[-\mu(\{f(\mathbf{s}) \not< \mathbf{z} \text{ or } f(\mathbf{t}) \not< \mathbf{y}\})]}{\exp[-\mu(\{f(\mathbf{t}) \not< \mathbf{y}\})]} \\
&= \exp[-\mu(\{f(\mathbf{s}) \not< \mathbf{z},\ f(\mathbf{t}) < \mathbf{y}\})].
\end{aligned}$$

□

3.5.4 Proof of Propositions 3.4.1 and 3.4.2

Proof of Proposition 3.4.1:
This is a straightforward application of Theorem 3.3.2 and 3.3.3. Take into account that when $K = \{t\}$, $\mathcal{P}_K$ is reduced to a unique partition of size $\ell = 1$ so that $\Theta = \{t\}$ and $\varphi_1^+ = \phi_t^+$. □

Proof of Proposition 3.4.2:
According to Proposition 3.4.1, $\mathbb{P}[\phi_t^+ \in \cdot \mid \eta(t) = y] = P_t(y, \cdot)$. For any measurable $A \subset \mathcal{C}_0$ and $B \subset (0, +\infty)$, we compute

$$\begin{aligned}
&\int_B \int_{\mathcal{C}_0} 1_{\{\frac{y}{f(t)} f \in A\}} f(t) \sigma(df) \mu_t(dy) \\
= &\int_0^\infty \int_{\mathcal{C}_0} \int_{\mathcal{C}_0} 1_{\{\frac{rg(t)}{f(t)} f \in A\}} 1_{\{rg(t) \in B\}} f(t) \sigma(df) r^{-2} dr \sigma(dg) \\
= &\int_{\mathcal{C}_0} \int_0^\infty 1_{\{rf \in A\}} 1_{\{rf(t) \in B\}} \, r^{-2} dr \sigma(df) \\
= &\int_{\mathcal{C}_0} 1_{\{f \in A\}} 1_{\{f(t) \in B\}} \, \mu(df)
\end{aligned}$$

The second equality follows from the change of variable $\tilde{r} = rg(t)/f(t)$ together with the relation $\int_{\mathcal{C}_0} g(t) \sigma(dg) = 1$. This proves that

$$P_t(y, A) = \int_{\mathcal{C}_0} 1_{\{\frac{y}{f(t)} f \in A\}} f(t) \sigma(df).$$

According to Equation (3.17)

$$\mathbb{P}[\eta(\mathbf{s}) < \mathbf{z} \mid \eta(t) = y] = P_t(y, \{f(\mathbf{s}) < \mathbf{z}\}) \exp[-\mu(\{f(\mathbf{s}) \not< \mathbf{z},\ f(t) < y\})].$$

We have

$$\begin{aligned}
P_t(y, \{f(\mathbf{s}) < \mathbf{z}\}) &= \int_{\mathcal{C}_0} 1_{\{\frac{y}{f(t)} f(\mathbf{s}) < \mathbf{z}\}} f(t) \sigma(df) \\
&= \int_{\mathcal{C}_0} 1_{\{\bigvee_{i=1}^l \frac{f(s_i)}{z_i} < \frac{f(t)}{y}\}} f(t) \sigma(df)
\end{aligned}$$

and

$$\begin{aligned}
\mu(\{f(\mathbf{s}) \not< \mathbf{z},\ f(t) < y\}) &= \int_0^\infty \int_{\mathcal{C}_0} 1_{\{rf(\mathbf{s}) \not< \mathbf{z}, rf(t) < y\}} r^{-2} dr \sigma(df) \\
&= \int_0^\infty \int_{\mathcal{C}_0} 1_{\{\min_{1 \le i \le l} \frac{z_i}{f(s_i)} \le r < \frac{y}{f(t)}\}} r^{-2} dr \sigma(df) \\
&= \int_{\mathcal{C}_0} \Big(\bigvee_{i=1}^l \frac{f(s_i)}{z_i} - \frac{f(t)}{y} \Big)^+ \sigma(df).
\end{aligned}$$

This proves Equation (3.19). □

3.A Appendix : Auxiliary results

3.A.1 Slyvniak's formula

Palm Theory deals with conditional distribution for point processes. We recall here one of the most famous formula of Palm theory, known as Slyvniak's Theorem. This will be the main tool in our computations. For a general reference on Poisson point processes, Palm theory and their applications, the reader is invited to refer to the monograph [74] by Stoyan, Kendall and Mecke.

Let $M_p(\mathcal{C}_0)$ be the set of locally-finite point measures N on $\mathcal{C}_0$ endowed with the σ-algebra generated by the family of mappings

$$N \mapsto N(A), \qquad A \subset \mathcal{C}_0 \text{ Borel set.}$$

Theorem 3.A.1 (Slyvniak's Formula)**.**
Let Φ be a Poisson point process on $\mathcal{C}_0$ with intensity measure μ. For any measurable function $F : \mathcal{C}_0^k \times M_p(\mathcal{C}_0) \to [0, +\infty)$,

$$\begin{aligned} &\mathbb{E}\Big[\int_{\mathcal{C}_0^k} F\Big(\phi_1, \ldots, \phi_k, \Phi - \sum_{i=1}^{k} \delta_{\phi_i}\Big)\, \Phi(d\phi_1)\, (\Phi - \delta_{\phi_1})(d\phi_2) \cdots \Big(\Phi - \sum_{j=1}^{k-1} \delta_{\phi_j}\Big)(d\phi_k)\Big] \\ = \;\; & \int_{\mathcal{C}_0^k} \mathbb{E}[F(f_1, \ldots, f_k, \Phi)]\, \mu^{\otimes k}(df_1, \ldots, df_k). \end{aligned}$$

3.A.2 Regular conditional distribution

We recall here briefly the notion of regular conditional probability (see e.g. Proposition A1.5.III in Daley and Vere-Jones [20]). Let $(\mathcal{Y}, \mathcal{G})$ be a complete separable metric space with its associated σ-algebra of Borel sets, $(\mathcal{X}, \mathcal{F})$ an arbitrary measurable space, and π a probability measure on the product space $(\mathcal{X} \times \mathcal{Y}, \mathcal{F} \otimes \mathcal{G})$. Let $\pi_{\mathcal{X}}$ denote the $\mathcal{X}$-marginal of π, i.e. $\pi_{\mathcal{X}}(A) = \pi(A \times \mathcal{Y})$ for any $A \in \mathcal{F}$. Then there exists a family of kernels $K(x, B)$ such that

- $K(x, \cdot)$ is a probability measure on $(\mathcal{Y}, \mathcal{G})$ for any fixed $x \in \mathcal{X}$;
- $K(\cdot, B)$ is an $\mathcal{F}$-measurable function on $\mathcal{X}$ for each fixed $B \in \mathcal{G}$;
- $\pi(A \times B) = \int_A K(x, B)\pi_{\mathcal{X}}(dx)$ for any $A \in \mathcal{F}$ and $B \in \mathcal{G}$.

These three properties define the notion of regular conditional probability. The existence of the regular conditional probability relies on the assumption that $\mathcal{Y}$ is a complete and separable metric space. Furthermore, for any $\mathcal{F} \otimes \mathcal{G}$-measurable nonnegative function f on $X \times Y$, it follows that

$$\int_{\mathcal{X} \times \mathcal{Y}} f(x, y)\pi(dx, dy) = \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y) K(x, dy) \pi_{\mathcal{X}}(dx).$$

The following Lemma states the existence of the kernel

$$\{P_{\mathbf{s}}(\mathbf{x}, df);\ \mathbf{x} \in [0, +\infty)^l \setminus \{0\}\}$$

satisfying Equation (3.12). This is not straightforward since the measure μ is not a probability measure and may be infinite.

Lemma 3.A.2. *The regular version of the conditional measure $\mu(df)$ with respect to $f(\mathbf{s}) \in [0,+\infty)^l \setminus \{0\}$ exists. It is denoted by $\{P_{\mathbf{s}}(\mathbf{x}, df);\ \mathbf{x} \in [0,+\infty)^l \setminus \{0\}\}$ and satisfies Equation* (3.12).

Proof :
Let $|\cdot|$ denote a norm on $[0,+\infty)^l$. Define $A = \{f \in \mathcal{C}_0;\ f(\mathbf{s}) \neq 0\}$ and, for $i \geq 0$, $A_i = \{f \in \mathcal{C}_0;\ (i+1)^{-1} \leq |f(\mathbf{s})| < i^{-1}\}$ with the convention $0^{-1} = +\infty$. Clearly, A is equal to the disjoint union of the A_i's. We note $\mu^i(\cdot) = \mu(\cdot \cap A_i)$ the measure on the complete and separable space $\mathcal{C}_0 \cup \{0\}$. Equation (3.4) ensures that μ_i is a finite measure (and hence a probability measure up to a normalization constant) and there exists a regular conditional probability kernel $P^i_{\mathbf{s}}(\mathbf{x}, df)$ with respect to $f(\mathbf{s}) = \mathbf{x}$. We obtain, for all $F : [0,+\infty)^l \times \mathcal{C}_0$,

$$\int_{A_i} F(f(\mathbf{s}), f)\mu(df) = \int_{\tilde{A}_i}\int_{\mathcal{C}_0} F(\mathbf{x}, f)P^i_{\mathbf{s}}(\mathbf{x}, df)\mu_{\mathbf{s}}(d\mathbf{x}),$$

where $\tilde{A}_i = \{x \in [0,+\infty)^k;\ (i+1)^{-1} \leq |\mathbf{x}| < i^{-1}\}$. Let us define $P_{\mathbf{s}}(\mathbf{x}, df)$ a probability measure on $\mathcal{C}_0$ by

$$P_{\mathbf{s}}(\mathbf{x}, df) = \sum_{i\geq 1} 1_{\{\mathbf{x}\in\tilde{A}_i\}} P^i_{\mathbf{s}}(\mathbf{x}, df).$$

If F vanishes on $\{0\} \times \mathcal{C}_0$, we obtain

$$\begin{aligned}
\int_{\mathcal{C}_0} F(f(\mathbf{s}), f)\mu(df) &= \sum_{i\geq 0}\int_{A_i} F(f(\mathbf{s}), f)\mu(df) \\
&= \sum_{i\geq 0}\int_{\tilde{A}_i}\int_{\mathcal{C}_0} F(\mathbf{x}, f)P^i_{\mathbf{s}}(\mathbf{x}, df)\mu_{\mathbf{s}}(d\mathbf{x}) \\
&= \int_{[0,+\infty)^k\setminus\{0\}}\int_{\mathcal{C}_0} F(\mathbf{x}, f)P_{\mathbf{s}}(\mathbf{x}, df)\mu_{\mathbf{s}}(d\mathbf{x}).
\end{aligned}$$

This proves Equation (3.12). □

3.A.3 Measurability properties

Lemma 3.A.3. *Φ^+_K and Φ^-_K are measurable from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(M_p(\mathcal{C}_0), \mathcal{M}_p)$.*

Proof :
From Definition 3.2.1, it is enough to prove that that the events $\{\phi_i <_K \eta\} \in \mathcal{F}$ and $\{\phi_i \not<_K \eta\}$ are $\mathcal{F}$-measurable. Let K_0 be a dense countable subset of K and note that $\phi <_K \eta$ if and only if there is some rational $\varepsilon > 0$ so that $\phi(t) < \eta(t) - \varepsilon$ for all $t \in K_0$. Hence, for all $n \in \mathbb{N} \cup \{+\infty\}$,

$$\begin{aligned}
\{\phi_i <_K \eta\,;\ N = n\} &= \bigcup_{\varepsilon>0}\bigcap_{t\in K_0}\{N = n\,;\ \phi_i(t) < \eta(t) - \varepsilon\} \\
&= \bigcup_{\varepsilon>0}\bigcap_{t\in K_0}\bigcup_{j\leq n}\{N = n\,;\ \phi_i(t) < \phi_j(t) - \varepsilon\} \qquad (3.24)
\end{aligned}$$

and $\{\phi_i <_K \eta\} = \bigcup_{n=0}^{\infty}\{\phi_i <_K \eta\ ;\ N = n\} \in \mathcal{F}$. Note the union over ε is countable. $\square$

Lemma 3.A.4. *The set* C_K^+ *and* $C_K^-(g)$ *are measurable in* $(M_p(\mathcal{C}_0), \mathcal{M}_p)$.

Proof :
Let g be a continuous function defined at least on K and consider the Borel set

$$A = \{f \in \mathcal{C}_0;\ f \not<_K g\} \subset \mathcal{C}_0.$$

The set $C_K^-(g)$ defined by Equation (3.6) is equal to

$$C_K^-(g) = \{M \in M_p(\mathcal{C}_0);\ M(A) = 0\}$$

and is $\mathcal{M}_p$-measurable.
In order to prove the measurability of C_K^+ defined by Equation (3.5), we introduce a measurable enumeration of the atoms of a point measure M (see Lemma 9.1.XIII in Daley and Vere-Jones [21]). Given an ordered dissecting system of $\mathcal{C}_0$, one can construct measurable applications

$$\kappa : M_p(\mathcal{C}_0) \to \mathbb{N} \cup \{\infty\} \quad \text{and} \quad \psi_i : M_p(\mathcal{C}_0) \to \mathcal{C}_0,\ i \geq 1,$$

such that

$$M = \sum_{i=1}^{\kappa(M)} \delta_{\psi_i(M)}, \quad M \in M_p(\mathcal{C}_0).$$

A point measure M does not lie in C_K^+ if and only if it has a K-subsextremal atom. Hence,

$$M_p(\mathcal{C}_0) \setminus C_K^+ = \bigcup_{k=0}^{+\infty} \bigcup_{i=1}^{k} \{\kappa(M) = k;\ \psi_i(M) <_K \max(M)\}.$$

Similar computations as in Equation (3.24) entail

$$\{\kappa(M) = k;\ \psi_i(M) <_K \max(M)\} = \bigcup_{\varepsilon > 0} \bigcap_{t \in K_0} \bigcup_{j \leq k} \{\kappa = k\ ;\ \psi_i(t) < \psi_j(t) - \varepsilon\},$$

whence C_K^+ is $\mathcal{M}_p$-measurable. $\square$

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Chapitre 4

Conditional simulation of max-stable processes

Abstract

Since many environmental processes are spatial in extent, a single extreme event may affect several locations, and their spatial dependence has to be appropriately taken into account. This chapter proposes a framework for conditional simulation of max-stable processes and gives closed forms for the regular conditional distributions of Brown–Resnick and Schlather processes. We test the method on simulated data and give an application to extreme rainfall around Zurich and extreme temperatures. The proposed framework provides accurate conditional simulations and can handle real-sized problems.

4.1 Introduction

Max-stable processes arise naturally when studying extremes of stochastic processes and therefore play a major role in the statistical modelling of spatial extremes [17, 58, 26]. Although a different spectral characterization of max-stable processes exists [30], for our purposes the most useful representation is [67]

$$Z(t) = \max_{i \geq 1} \zeta_i Y_i(t), \qquad t \in \mathbb{R}^d, \tag{4.1}$$

where $\{\zeta_i\}_{i \geq 1}$ are the points of a Poisson process on $(0, \infty)$ with intensity $\zeta^{-2} \mathrm{d}\zeta$ and the Y_i are independent replicates of a nonnegative stochastic process Y such that $E\{Y(t)\} = 1$ for all $t \in \mathbb{R}^d$. It is well known that Z is a max-stable process on $\mathbb{R}^d$ with unit Fréchet margins [31, 67]. Although (4.1) takes the pointwise maximum over an infinite number of points $\{\zeta_i\}$ and processes Y_i, it is possible to get approximate realizations from Z [67, 57].

Based on (4.1) several parametric max-stable models have been proposed [67, 16, 51, 26] that share the same finite dimensional distribution functions

$$\mathrm{pr}\{Z(t_1) \leq z_1, \ldots, Z(t_k) \leq z_k\} = \exp\left[-E\left\{\max_{j=1,\ldots,k} \frac{Y(t_j)}{z_j}\right\}\right],$$

where $k \in \mathbb{N}$, $z_1, \dots, z_k > 0$ and $t_1, \dots, t_k \in \mathbb{R}^d$.

Apart from the Smith model [42], only the bivariate cumulative distribution functions are explicitly known. To bypass this hurdle, de Haan and Pereira [32] propose a semi-parametric estimator and Padoan et al. [58] suggest the use of the maximum pairwise likelihood estimator.

Paralleling the use of the variogram in classical geostatistics, the extremal coefficient function [69, 19]

$$\theta(t_1 - t_2) = -z \log \mathrm{pr}\{Z(t_1) \leq z, Z(t_2) \leq z\}$$

is widely used to summarize the spatial dependence of extremes for stationary processes. It takes values in the interval $[1, 2]$; the lower bound indicates complete dependence, and the upper one independence.

The last decade has seen many advances to develop geostatistics for extremes, and software is available to practitioners [78, 68, 64]. However an important tool currently missing is conditional simulation of max-stable processes. In classical geostatistic based on Gaussian models, conditional simulations are well established [18] and provide a framework to assess the distribution of a Gaussian random field given values observed at fixed locations. For example, conditional simulations of Gaussian processes have been used to model land topography [55].

Conditional simulation of max-stable processes is a long-standing problem [23, 24]. Wang and Stoev [79] provide a first solution, but their framework is limited to processes having a discrete spectral measure and thus may be too restrictive to appropriately model the spatial dependence in complex situations.

Based on the recent developments on the regular conditional distribution of max-infinitely divisible processes, the aim of this chapter is to provide a methodology for conditional simulation of max-stable processes with continuous spectral measures. More formally for a study region $\mathcal{X} \subset \mathbb{R}^d$, our goal is to derive an algorithm to sample from the regular conditional distribution of $Z \mid \{Z(t_1) = z_1, \dots, Z(t_k) = z_k\}$ for some $z_1, \dots, z_k > 0$ and k conditioning locations $t_1, \dots, t_k \in \mathcal{X}$.

4.2 Conditional simulation of max-stable processes

4.2.1 General framework

This section reviews some key results of an unpublished paper available from the first author with a particular emphasis on max-stable processes. Our goal is to give a more practical interpretation of their results from a simulation perspective. To this aim, we recall two key results and propose a procedure to get conditional realizations of max-stable processes.

Let $\mathbb{R}^{\mathcal{X}}$ be the space of real-valued functions on $\mathcal{X} \subset \mathbb{R}^d$ and let $\Phi = \{\varphi_i\}_{i \geq 1}$ be a Poisson point process on $\mathbb{R}^{\mathcal{X}}$ where $\varphi_i(t) = \zeta_i Y_i(t)$ $(i = 1, 2, \dots)$ with ζ_i and Y_i as in (4.1). We write $f(t) = \{f(t_1), \dots, f(t_k)\}$ for all random functions $f \colon \mathcal{X} \to \mathbb{R}$ and $t = (t_1, \dots, t_k) \in \mathcal{X}^k$. It is not difficult to show that for all Borel sets $A \subset \mathbb{R}^k$, the Poisson process $\{\varphi_i(t)\}_{i \geq 1}$ defined on $\mathbb{R}^k$ has intensity measure

$$\mu_t(A) = \int_0^\infty \mathrm{pr}\{\zeta Y(t) \in A\}\, \zeta^{-2} \mathrm{d}\zeta.$$

The point process Φ is called regular if the intensity measure μ_t has an intensity function λ_t, i.e., $\mu_t(\mathrm{d}z) = \lambda_t(z)\mathrm{d}z$, for all $t \in \mathcal{X}^k$.

The first key result is that provided the Poisson process Φ is regular, the intensity function λ_t and the conditional intensity function

$$\lambda_{s|t,z}(u) = \frac{\lambda_{(s,t)}(u,z)}{\lambda_t(z)}, \qquad (s,t) \in \mathcal{X}^{m+k}, \qquad u \in \mathbb{R}^m, \qquad z \in (0,+\infty)^k, \tag{4.2}$$

drives how the conditioning terms $\{Z(t_j) = z_j\}$ $(j = 1,\dots,k)$ are met, see Steps 1 and 2 in Theorem 4.2.1.

The second key result is that, conditionally on $Z(t) = z$, the Poisson process Φ can be decomposed into two independent point processes, say $\Phi = \Phi^- \cup \Phi^+$, where

$$\Phi^- = \{\varphi \in \Phi \colon \varphi(t_i) < z_i,\ i = 1,\dots,k\}, \qquad \Phi^+ = \bigcup_{i=1}^{k} \{\varphi \in \Phi \colon \varphi(t_i) = z_i\}.$$

Before introducing a procedure to get conditional realizations of max-stable processes, we introduce notation and make connections with the pioneering work of Wang and Stoev [79].

A function $\varphi \in \Phi^+$ such that $\varphi(t_i) = z_i$ for some $i \in \{1,\dots,k\}$ is called an extremal function associated to t_i and denoted by $\varphi^+_{t_i}$. It is easy to show that there exists almost surely a unique extremal function associated to t_i. Although $\Phi^+ = \{\varphi^+_{t_1},\dots,\varphi^+_{t_k}\}$ almost surely, it might happen that a single extremal function contributes to the random vector $Z(t)$ at several locations t_i, e.g., $\varphi^+_{t_1} = \varphi^+_{t_2}$. To take such repetitions into account, we define a random partition $\theta = (\theta_1,\dots,\theta_\ell)$ of the set $\{t_1,\dots,t_k\}$ into $\ell = |\theta|$ blocks and extremal functions $(\varphi^+_1,\dots,\varphi^+_\ell)$ such that $\Phi^+ = \{\varphi^+_1,\dots,\varphi^+_\ell\}$ and $\varphi^+_j(t_i) = z_i$ if $t_i \in \theta_j$ and $\varphi^+_j(t_i) < z_i$ if $t_i \notin \theta_j$ $(i = 1,\dots,k;\ j = 1,\dots,\ell)$. Wang and Stoev [79] call the partition θ the hitting scenario. The set of all possible partitions of $K = \{t_1,\dots,t_k\}$, denoted $\mathcal{P}_K$, identifies all possible hitting scenarios.

From a simulation perspective, the fact that Φ^- and Φ^+ are independent given $Z(t) = z$ is especially convenient and suggests a three–step procedure to sample from the conditional distribution of Z given $Z(t) = z$.

Theorem 4.2.1. *Suppose that the point process Φ is regular and let $(t,s) \in \mathcal{X}^{k+m}$. For $\tau = (\tau_1,\dots,\tau_\ell) \in \mathcal{P}_K$ and $j = 1,\dots,\ell$, define $I_j = \{i \colon t_i \in \tau_j\}$, $t_{\tau_j} = (t_i)_{i \in I_j}$, $z_{\tau_j} = (z_i)_{i \in I_j}$, $t_{\tau_j^c} = (t_i)_{i \notin I_j}$ and $z_{\tau_j^c} = (z_i)_{i \notin I_j}$. Consider the three–step procedure :*

step 1 : Draw a random partition $\theta \in \mathcal{P}_K$ with distribution

$$\pi_t(z,\tau) = \mathrm{pr}\left\{\theta = \tau \mid Z(t) = z\right\} = \frac{1}{C(t,z)} \prod_{j=1}^{|\tau|} \lambda_{t_{\tau_j}}(z_{\tau_j}) \int_{\{u_j < z_{\tau_j^c}\}} \lambda_{t_{\tau_j^c}|t_{\tau_j},z_{\tau_j}}(u_j) du_j,$$

where the normalization constant is

$$C(t,z) = \sum_{\tilde\tau \in \mathcal{P}_K} \prod_{j=1}^{|\tilde\tau|} \lambda_{t_{\tilde\tau_j}}(z_{\tilde\tau_j}) \int_{\{u_j < z_{\tilde\tau_j^c}\}} \lambda_{t_{\tilde\tau_j^c}|t_{\tilde\tau_j},z_{\tilde\tau_j}}(u_j) du_j.$$

step 2 : Given $\tau = (\tau_1, \dots, \tau_\ell)$, draw ℓ independent random vectors $\varphi_1^+(s), \dots, \varphi_\ell^+(s)$ from the distribution

$$\mathrm{pr}\left\{\varphi_j^+(s) \in dv \mid Z(t) = z, \theta = \tau\right\} = \frac{1}{C_j}\left\{\int 1_{\{u<z_{\tau_j^c}\}}\lambda_{(s,t_{\tau_j^c})|t_{\tau_j},z_{\tau_j}}(v,u)du\right\} dv$$

where $1_{\{\cdot\}}$ is the indicator function and

$$C_j = \int 1_{\{u<z_{\tau_j^c}\}}\lambda_{(s,t_{\tau_j^c})|t_{\tau_j},z_{\tau_j}}(v,u)dudv,$$

and define the random vector $Z^+(s) = \max_{j=1,\dots,\ell}\varphi_j^+(s)$.

step 3 : Independently draw a Poisson point process $\{\zeta_i\}_{i\geq 1}$ on $(0,\infty)$ with intensity $\zeta^{-2}d\zeta$ and $\{Y_i\}_{i\geq 1}$ independent copies of Y, and define the random vector

$$Z^-(s) = \max_{i\geq 1}\zeta_i Y_i(s) 1_{\{\zeta_i Y_i(t) < z\}}.$$

Then the random vector $\tilde{Z}(s) = \max\{Z^+(s), Z^-(s)\}$ follows the conditional distribution of $Z(s)$ given $Z(t) = z$.

The corresponding conditional cumulative distribution function is

$$\mathrm{pr}\left\{Z(s) \leq a \mid Z(t) = z\right\} = \left\{\sum_{\tau\in\mathcal{P}_K}\pi_t(z,\tau)\prod_{j=1}^{|\tau|}F_{\tau,j}(a)\right\}\frac{\mathrm{pr}\{Z(s)\leq a, Z(t)\leq z\}}{\mathrm{pr}\{Z(t)\leq z\}}, \qquad (4.3)$$

where

$$F_{\tau,j}(a) = \mathrm{pr}\left\{\varphi_j^+(s) \leq a \mid Z(t) = z,\, \theta = \tau\right\} = \frac{\int_{\{u<z_{\tau_j^c}, v<a\}}\lambda_{(s,t_{\tau_j^c})|t_{\tau_j},z_{\tau_j}}(v,u)\mathrm{d}u\mathrm{d}v}{\int_{\{u<z_{\tau_j^c}\}}\lambda_{t_{\tau_j^c}|t_{\tau_j},z_{\tau_j}}(u)\mathrm{d}u}.$$

The first term in the right hand side of (4.3) corresponds to Steps 1 and 2 while the ratio is a consequence of Step 3. It is clear from (4.3) that the conditional random field $Z \mid \{Z(t) = z\}$ is not max-stable.

4.2.2 Distribution of the extremal functions

In this section we derive closed forms for the intensity function $\lambda_t(z)$ and the conditional intensity function $\lambda_{s|x,z}(u)$ for two widely used max-stable processes ; the Brown–Resnick [16, 51] and the Schlather [67] processes. Details of the derivations are given in the Appendix.

The Brown–Resnick process corresponds to the case where $Y(t) = \exp\{W(t) - \gamma(t)\}$ $(t \in \mathbb{R}^d)$ in (4.1) with W a centered Gaussian process with stationary increments, semi variogram γ and such that $W(0) = 0$ almost surely. For $t \in \mathcal{X}^k$ and provided the covariance matrix Σ_t of the random vector $W(t)$ is positive definite, the intensity function is

$$\lambda_t(z) = C_t\exp\left(-\frac{1}{2}\log z^T Q_t \log z + L_t\log z\right)\prod_{i=1}^k z_i^{-1}, \qquad z\in(0,\infty)^k,$$

with $1_k = (1)_{i=1,\ldots,k}$, $\gamma_t = \{\gamma(t_i)\}_{i=1,\ldots,k}$,

$$Q_t = \Sigma_t^{-1} - \frac{\Sigma_t^{-1} 1_k 1_k^T \Sigma_t^{-1}}{1_k^T \Sigma_t^{-1} 1_k}, \qquad L_t = \left(\frac{1_k^T \Sigma_t^{-1} \gamma_t - 1}{1_k^T \Sigma_t^{-1} 1_k} 1_k - \gamma_t \right)^T \Sigma_t^{-1},$$

$$C_t = (2\pi)^{(1-k)/2} |\Sigma_t|^{-1/2} (1_k^T \Sigma_t^{-1} 1_k)^{-1/2} \exp\left\{ \frac{1}{2} \frac{(1_k^T \Sigma_t^{-1} \gamma_t - 1)^2}{1_k^T \Sigma_t^{-1} 1_k} - \frac{1}{2} \gamma_t^T \Sigma_t^{-1} \gamma_t \right\},$$

and for all $(s,t) \in \mathcal{X}^{m+k}$, $(u,z) \in (0,\infty)^{m+k}$ and provided the covariance matrix $\Sigma_{(s,t)}$ is positive definite, the conditional intensity function corresponds to a multivariate log-normal probability density function

$$\lambda_{s|t,z}(u) = (2\pi)^{-m/2} |\Sigma_{s|t}|^{-1/2} \exp\left\{ -\frac{1}{2} (\log u - \mu_{s|t,z})^T \Sigma_{s|t}^{-1} (\log u - \mu_{s|t,z}) \right\} \prod_{i=1}^m u_i^{-1},$$

where $\mu_{s|t,z} \in \mathbb{R}^m$ and $\Sigma_{s|t}$ are the mean and covariance matrix of the underlying normal distribution and are given by

$$\Sigma_{s|t}^{-1} = J_{m,k}^T Q_{(s,t)} J_{m,k}, \qquad \mu_{s|t,z} = \left\{ L_{(s,t)} J_{m,k} - \log z^T \tilde{J}_{m,k}^T Q_{(s,t)} J_{m,k} \right\} \Sigma_{s|t},$$

with

$$J_{m,k} = \begin{bmatrix} \mathrm{I}_m \\ 0_{k,m} \end{bmatrix}, \qquad \tilde{J}_{m,k} = \begin{bmatrix} 0_{m,k} \\ \mathrm{I}_k \end{bmatrix},$$

where I_k denotes the $k \times k$ identity matrix and $0_{m,k}$ the $m \times n$ null matrix.

The Schlather process considers the case $Y(t) = (2\pi)^{1/2} \max\{0, \varepsilon(t)\}$ $(t \in \mathbb{R}^d)$ in (4.1) with ε a standard Gaussian process with correlation function ρ. The associated point process Φ is not regular and it is more convenient to consider the equivalent representation where $Y(t) = (2\pi)^{1/2} \varepsilon(t)$ $(t \in \mathbb{R}^d)$. For $t \in \mathcal{X}^k$ and provided the covariance matrix Σ_t of the random vector $\varepsilon(t)$ is positive definite, the intensity function is

$$\lambda_t(z) = \pi^{-(k-1)/2} |\Sigma_t|^{-1/2} a_t(z)^{-(k+1)/2} \Gamma\left(\frac{k+1}{2} \right), \qquad z \in \mathbb{R}^k,$$

where $a_t(z) = z^T \Sigma_t^{-1} z$.

For $(s,t) \in \mathcal{X}^{m+k}$, $(u,z) \in \mathbb{R}^{m+k}$ and provided that the covariance matrix $\Sigma_{(s,t)}$ is positive definite, the conditional intensity function $\lambda_{s|t,z}(u)$ corresponds to the density of a multivariate Student distribution with $k+1$ degrees of freedom, location parameter $\mu = \Sigma_{s:t} \Sigma_t^{-1} z$, and scale matrix

$$\tilde{\Sigma} = \frac{a_t(z)}{k+1} \left(\Sigma_s - \Sigma_{s:t} \Sigma_t^{-1} \Sigma_{t:s} \right), \qquad \Sigma_{(s,t)} = \begin{bmatrix} \Sigma_s & \Sigma_{s:t} \\ \Sigma_{t:s} & \Sigma_t \end{bmatrix}.$$

4.3 Markov chain Monte Carlo sampler

The previous section introduced a procedure to get realizations from the regular conditional distribution of max-stable processes. This sampling scheme amounts to sampling from a discrete distribution whose state space corresponds to all possible partitions of the set of conditioning points, see Theorem 4.2.1 Step 1. Hence, even for a moderate number k of conditioning locations, the state space $\mathcal{P}_K$ becomes very large and the distribution $\pi_t(z, \cdot)$ cannot be computed. A Gibbs sampler is especially convenient.

For $\tau \in \mathcal{P}_K$, let τ_{-j} be the restriction of τ to the set $\{t_1, \dots, t_k\} \setminus \{t_j\}$. Our goal is to simulate from the conditional distribution

$$\mathrm{pr}(\theta \in \cdot \mid \theta_{-j} = \tau_{-j}), \tag{4.4}$$

where $\theta \in \mathcal{P}_K$ is a random partition which follows the target distribution $\pi_t(z, \cdot)$.

Since the number of possible updates is always less than k, a combinatorial explosion is avoided. Indeed for $\tau \in \mathcal{P}_K$ of size ℓ, the number of partitions $\tau^* \in \mathcal{P}_K$ such that $\tau^*_{-j} = \tau_{-j}$ for some $j \in \{1, \dots, k\}$ is

$$b^+ = \begin{cases} \ell & \text{if } \{t_j\} \text{ is a partitioning set of } \tau, \\ \ell + 1 & \text{if } \{t_j\} \text{ is not a partitioning set of } \tau, \end{cases}$$

since the point t_j may be reallocated to any partitioning set of τ_{-j} or to a new one.

To illustrate consider the set $\{t_1, t_2, t_3\}$ and let $\tau = (\{t_1, t_2\}, \{t_3\})$. Then the possible partitions τ^* such that $\tau^*_{-2} = \tau_{-2}$ are $(\{t_1, t_2\}, \{t_3\})$, $(\{t_1\}, \{t_2\}, \{t_3\})$, $(\{t_1\}, \{t_2, t_3\})$, while there exists only two partitions such that $\tau^*_{-3} = \tau_{-3}$, i.e., $(\{t_1, t_2\}, \{t_3\})$, $(\{t_1, t_2, t_3\})$.

The distribution (4.4) has nice properties. Since for all $\tau^* \in \mathcal{P}_K$ such that $\tau^*_{-j} = \tau_{-j}$ we have

$$\mathrm{pr}\left(\theta = \tau^* \mid \theta_{-j} = \tau_{-j}\right) = \frac{\pi_t(z, \tau^*)}{\displaystyle\sum_{\tilde{\tau} \in \mathcal{P}_K} \pi_t(z, \tilde{\tau}) 1_{\{\tilde{\tau}_{-j} = \tau_{-j}\}}} \propto \frac{\prod_{j=1}^{|\tau^*|} w_{\tau^*,j}}{\prod_{j=1}^{|\tau|} w_{\tau,j}}, \tag{4.5}$$

where

$$w_{\tau,j} = \lambda_{t_{\tau_j}}(z_{\tau_j}) \int_{\{u < z_{\tau_j^c}\}} \lambda_{t_{\tau_j^c} | t_{\tau_j}, z_{\tau_j}}(u) \mathrm{d}u.$$

Since many factors cancel out on the right hand side of (4.5), the Gibbs sampler is especially convenient.

The most computationally demanding part of (4.5) is the evaluation of the integral

$$\int_{\{u < z_{\tau_j^c}\}} \lambda_{t_{\tau_j^c} | t_{\tau_j}, z_{\tau_j}}(u) \mathrm{d}u.$$

For the Brown–Resnick and Schlather processes, we follow the lines of Genz [43] and compute these probabilities using a separation of variables method which provides a transformation of the original integration problem to the unit hyper-cube. A quasi Monte Carlo scheme and antithetic variable sampling are used to improve efficiency.

Since it is not obvious how to implement a Gibbs sampler whose target distribution has support $\mathcal{P}_K$, the remainder of this section gives practical details. For any fixed locations $t_1, \dots, t_k \in \mathcal{X}$, we first describe how each partition of $\{t_1, \dots, t_k\}$ is stored. To illustrate consider the set $\{t_1, t_2, t_3\}$ and the partition $(\{t_1, t_2\}; \{t_3\})$. This partition is defined as $(1, 1, 2)$, indicating that t_1 and t_2 belong to the same partitioning set labeled 1 and t_3 belongs to the partitioning set 2. There exists several equivalent notations for this partition : for example one can use $(2, 2, 1)$ or $(1, 1, 3)$. Since there is a one-one mapping between $\mathcal{P}_K$ and the set

$$\mathcal{P}_K^* = \left\{(a_1, \ldots, a_k)\colon i \in \{2, \ldots, k\},\ 1 = a_1 \leq a_i \leq \max_{1 \leq j < i} a_j + 1,\ a_i \in \mathbb{Z}\right\},$$

we shall restrict our attention to the partitions that live in $\mathcal{P}_K^*$ and going back to our example we see that $(1, 1, 2)$ is valid but $(2, 2, 1)$ and $(1, 1, 3)$ are not.

For $\tau \in \mathcal{P}_K^*$ of size ℓ, let $r_1 = \sum_{i=1}^k 1_{\{\tau_i = a_j\}}$ and $r_2 = \sum_{i=1}^k 1_{\{\tau_i = b\}}$, i.e., the number of conditioning locations that belong to the partitioning sets a_j and b where $b \in \{1, \ldots, b^+\}$ with

$$b^+ = \begin{cases} \ell & (r_1 = 1), \\ \ell + 1 & (r_1 \neq 1). \end{cases}$$

Then the conditional probability distribution (4.5) satisfies

$$\operatorname{pr}(\tau_j = b \mid \tau_i = a_i,\ i \neq j) \propto \begin{cases} 1 & (b = a_j), & \text{(4.6a)} \\ w_{\tau^*,b}/(w_{\tau,b} w_{\tau,a_j}) & (r_1 = 1, r_2 \neq 0, b \neq a_j), & \text{(4.6b)} \\ w_{\tau^*,b} w_{\tau^*,a_j}/(w_{\tau,b} w_{\tau,a_j}) & (r_1 \neq 1, r_2 \neq 0, b \neq a_j), & \text{(4.6c)} \\ w_{\tau^*,b} w_{\tau^*,a_j}/w_{\tau,a_j} & (r_1 \neq 1, r_2 = 0, b \neq a_j), & \text{(4.6d)} \end{cases}$$

where $\tau^* = (a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_k)$. Although τ^* may not belong to $\mathcal{P}_K^*$, it corresponds to a unique partition of $\mathcal{P}_K$ and we can use the bijection $\mathcal{P}_K \to \mathcal{P}_K^*$ to recode τ^* into an element of $\mathcal{P}_K^*$. In (4.6a)–(4.6d) the event $\{r_1 = 1, r_2 = 0, b \neq a_j\}$ is missing since $\{r_1 = 1, r_2 = 0\}$ implies that $\tau^* = \tau$, where the equality has to be understood in terms of elements of $\mathcal{P}_K$, and this case has been already taken into account with (4.6a).

Once these conditional weights have been computed, the Gibbs sampler proceeds by updating each element of τ successively. We use a random scan implementation of the Gibbs sampler [54]. More precisely, one iteration of the random scan Gibbs sampler selects an element of τ at random according to a given distribution, say $p = (p_1, \ldots, p_k)$, and then updates this element. Throughout this chapter we will use the uniform random scan Gibbs sampler for which the selection distribution is assumed to be a discrete uniform distribution, i.e., $p = (k^{-1}, \ldots, k^{-1})$.

4.4 Simulation Study

In this section we check if our algorithm is able to produce realistic conditional simulations of Brown–Resnick and Schlather processes. For each model, we consider three different sample path properties, as summarized in Table 4.1. These configurations were chosen such that the spatial dependence structures are similar to our applications in Section 4.5.

In order to check if our sampling procedure is accurate and given a single conditional event $\{Z(t) = z\}$ for each configuration, we generated 1000 conditional realizations with standard

TABLE 4.1 – Sample path properties of the max-stable models. For the Brown–Resnick model, the variogram parameters are set to ensure that the extremal coefficient function satisfies $\theta(115) = 1{\cdot}7$ while the correlation function parameters are set to ensure that $\theta(100) = 1{\cdot}5$ for the Schlather model.

	Brown–Resnick : $\gamma(h) = (h/\lambda)^\kappa$			Schlather : $\rho(h) = \exp\{-(h/\lambda)^\kappa\}$		
	γ_1 : Very wiggly	γ_2 : Wiggly	γ_3 : Smooth	ρ_1 : Very wiggly	ρ_2 : Wiggly	ρ_3 : Smooth
λ	25	54	69	208	144	128
κ	0·5	1·0	1·5	0·5	1·0	1·5

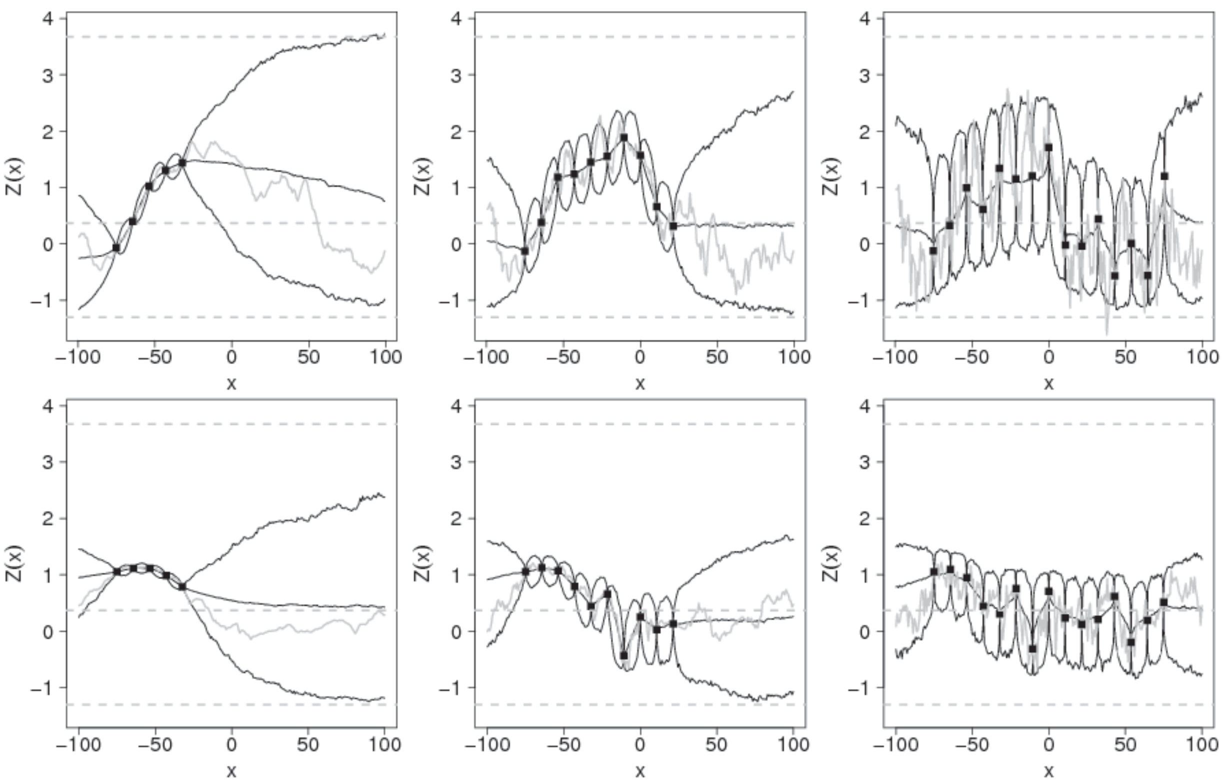

FIGURE 4.1 – Pointwise sample quantiles estimated from 1000 conditional simulations of max-stable processes with standard Gumbel margins with $k = 5, 10, 15$ conditioning locations. The top row shows results for the Brown–Resnick models with semi variograms $\gamma_3, \gamma_2, \gamma_1$, from left to right. The bottom row shows results for the Schlather models with correlation functions ρ_3, ρ_2, ρ_1, from left to right. The solid black lines shows the pointwise 0·025, 0·5, 0·975 sample quantiles and the dashed grey lines that of a standard Gumbel distribution. The squares show the conditional points $\{(t_i, z_i)\}_{i=1,\ldots,k}$. The solid grey lines correspond the simulated paths used to get the conditioning events.

TABLE 4.2 – Computational timings for conditional simulations of max-stable processes on a 50×50 grid defined on the square $[0, 100 \times 2^{1/2}]^2$ for a varying number k of conditioning locations uniformly distributed over the region. The timings, in seconds, are mean values over 100 conditional simulations ; standard deviations are reported in brackets.

	Brown–Resnick : $\gamma(h) = (h/25)^{0\cdot5}$				Schlather : $\rho(h) = \exp\left\{-(h/208)^{0\cdot50}\right\}$			
	Step 1	Step 2	Step 3	Overall	Step 1	Step 2	Step 3	Overall
$k = 5$	0·21 (0·01)	49 (11)	1·4 (0·1)	50 (11)	1·40 (0·02)	1·9 (0·7)	0·9 (0·3)	4·2 (0·8)
$k = 10$	8 (2)	76 (18)	1·4 (0·1)	85 (19)	12 (4)	2·4 (0·8)	1·0 (0·3)	15 (4)
$k = 25$	95 (38)	117 (30)	1·4 (0·1)	214 (61)	86 (42)	4 (1)	1·0 (0·3)	90 (43)
$k = 50$	583 (313)	348 (391)	1·5 (0·1)	931 (535)	367 (222)	62 (113)	1·0 (0·3)	430 (262)

Conditional simulations with $k = 5$ do not use a Gibbs sampler.

Gumbel margins. Figure 4.1 shows the pointwise sample quantiles obtained from these 1000 simulated paths and compares them to unit Gumbel quantiles. As expected the conditional sample paths inherit the regularity driven by the shape parameter κ and there is less variability in regions close to conditioning locations. Since the Brown–Resnick processes considered are ergodic [50], for regions far away from any conditioning location the sample quantiles converges to that of a standard Gumbel distribution indicating that the conditional event has no influence. This is not the case for the non-ergodic Schlather processes. Most of the time the sample paths used to get the conditional events belong to the 95% pointwise confidence intervals, corroborating that our sampling procedure seems to be accurate.

Table 4.2 gives computational timings for conditional simulations of max-stable processes on a 50×50 grid with a varying number of conditioning locations. Due to the combinatorial complexity of the partition set $\mathcal{P}_K$, the timings increase rapidly with respect to the number of conditioning points k. It is however reassuring that the algorithm is tractable when $k \in \{1, \dots, 50\}$; hence covering many practical situations and applications.

4.5 Application

4.5.1 Extreme precipitation around Zurich

The data considered here were previously analyzed by Davison et al. [26] who showed that Brown–Resnick processes were among the best models for these data. The data are summer maximum rainfall for the years 1962–2008 at 51 weather stations in the Plateau region of Switzerland, provided by the national meteorological service, MeteoSuisse. To ensure strong dependence between the conditioning locations, we consider as conditional locations the 24 weather stations that are at most 30km distant from Zurich and set as the conditional values the rainfall amounts recorded in the year 2000, the year of the largest precipitation event ever recorded in Zurich between 1962–2008, see Figure 4.2. The largest and smallest distances between the conditioning locations are around 55km and just over 4km respectively.

A Brown–Resnick process having semi variogram $\gamma(h) = (h/\lambda)^\kappa$ has to be fitted and the maximum pairwise likelihood estimator introduced by Padoan et al. [58] was used to simulta-

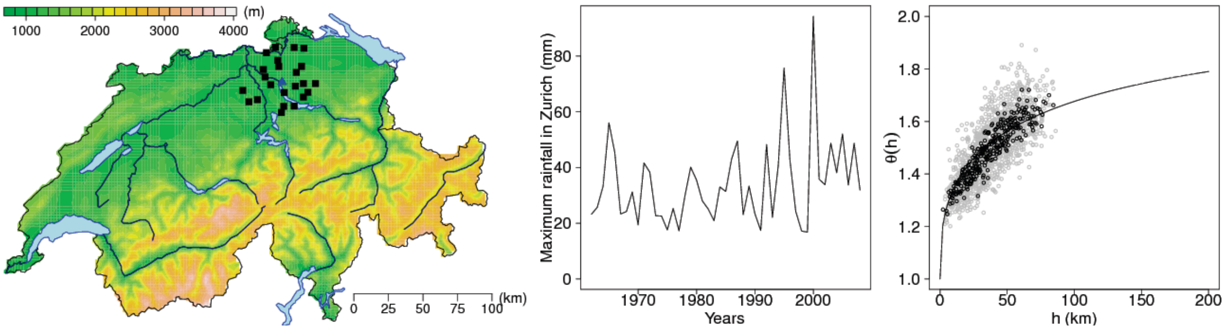

FIGURE 4.2 – Left : Map of Switzerland showing the stations of the 24 rainfall gauges used for the analysis, with an insert showing the altitude. The station marked with a triangle corresponds to Zurich. Middle : Summer maximum rainfall values for 1962–2008 at Zurich. Right : Comparison between the pairwise extremal coefficient estimates for the 51 original weather stations and the extremal coefficient function derived from a fitted Brown–Resnick processes having semi variogram $\gamma(h) = (h/\lambda)^{\kappa}$. The grey points are pairwise estimates ; the black ones are binned estimates and the curve is the fitted extremal coefficient function.

TABLE 4.3 – Distribution of the partition size for the rainfall data estimated from a simulated Markov chain of length 15000

Partition size	1	2	3	4	5	6	7–24
Empirical probabilities (%)	66·2	28·0	4·8	0·5	0·2	0·2	<0·05

neously fit the marginal parameters and the spatial dependence parameters λ and κ. In accordance with Davison et al. [26], the marginal parameters were described by $\eta(t) = \beta_{0,\eta} + \beta_{1,\eta}\mathrm{lon}(t) + \beta_{2,\eta}\mathrm{lat}(t)$, $\sigma(t) = \beta_{0,\sigma} + \beta_{1,\sigma}\mathrm{lon}(t) + \beta_{2,\sigma}\mathrm{lat}(t)$, $\xi(t) = \beta_{0,\xi}$, where $\eta(t), \sigma(t), \xi(t)$ are the location, scale and shape parameters of the generalized extreme value distribution and $\mathrm{lon}(t), \mathrm{lat}(t)$ the longitude and latitude of the stations at which the data are observed. The maximum pairwise likelihood estimates for λ and κ are 38 (14) and 0·69 (0·07) and give a practical extremal range, i.e., the distance h_+ such that $\theta(h_+) = 1{\cdot}7$, of around 115km, see the right panel of Figure 4.2.

Table 4.3 shows the distribution of the partition size estimated from a Markov chain of length 15000. Around 65% of the time the summer maxima observed at the 24 conditioning locations were a consequence of a single extremal function, i.e., only one storm event, and around 30% of the time a consequence of two different storms. Since the simulated Markov chain keeps a trace of all the simulated partitions, we looked at the partitions of size two and saw that around 65% of the time, at least one of the four up–north conditioning locations was impacted by one extremal function while the remaining 20 locations were always influenced by another one.

Figure 4.3 plots the pointwise 0·025, 0·5 and 0·975 sample quantiles obtained from 10000 conditional simulations of our fitted Brown–Resnick process. The conditional median provides a point estimate for the rainfall at an ungauged location and the 0·025 and 0·975 conditional quantiles a 95% pointwise confidence interval. As indicated by Figure 4.1, the shape parameter κ has a major impact on the regularity of paths and on the width of the confidence interval. The value $\hat{\kappa} \approx 0{\cdot}69$ corresponds to very wiggly sample paths and wider confidence intervals. To

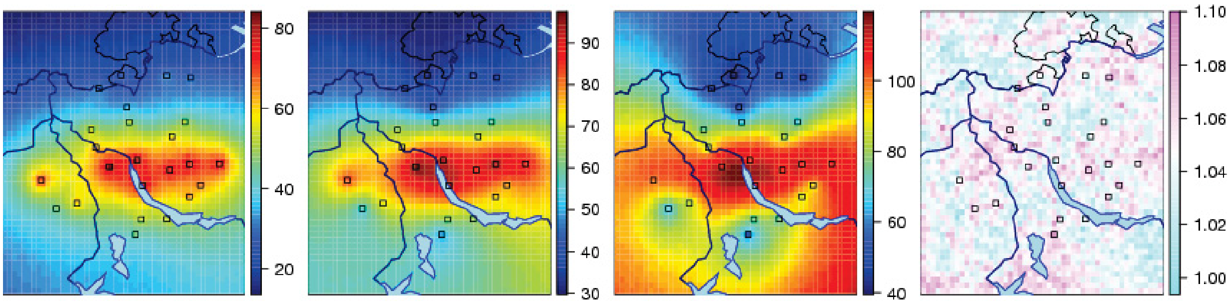

FIGURE 4.3 – From left to right, maps on a 50×50 grid of the pointwise 0·025, 0·5 and 0·975 sample quantiles for rainfall (mm) obtained from 10000 conditional simulations of Brown–Resnick processes having semi variogram $\gamma(h) = (h/38)^{0 \cdot 69}$. The rightmost panel plots the ratio of the pointwise confidence intervals with and without taking into account the parameter estimate uncertainties. The squares show the conditional locations.

assess the impact of parameter uncertainties on conditional simulations, the ratio of the width of the confidence intervals with or without parameter uncertainty is shown in the right panel of Figure 4.3. The uncertainties were taken into account by sampling from the asymptotic distribution of the maximum composite likelihood estimator and draw one conditional simulation for each realization. These ratios show no clear spatial pattern and the width of the confidence interval is increased by at most 10%.

4.5.2 Extreme temperatures in Switzerland

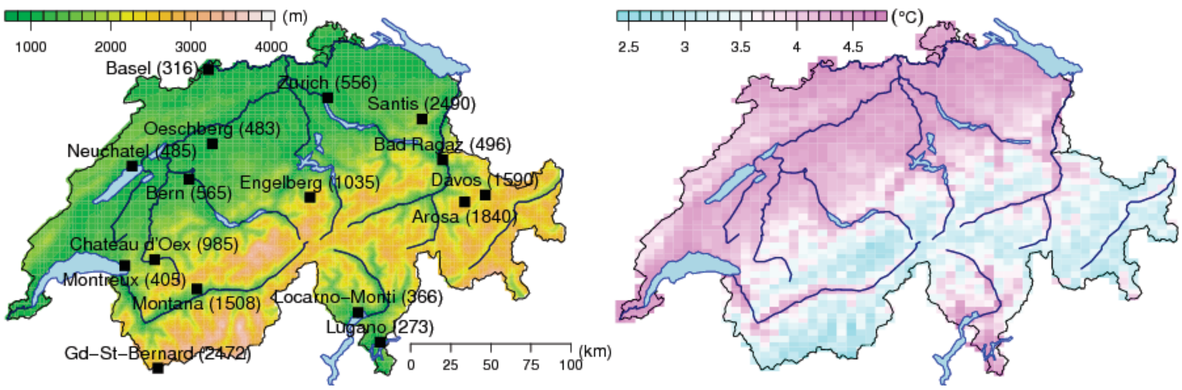

FIGURE 4.4 – Left : Topographical map of Switzerland showing the sites and altitudes in metres above sea level of 16 weather stations for which annual maxima temperature data are available. Right : Map of temperature anomalies ($^\circ$C), i.e., the difference between the pointwise conditional obtained from 10000 conditional simulations and unconditional medians estimated from the fitted Schlather process.

The data considered here consist in annual maximum temperatures recorded at 16 sites in Switzerland during the period 1961–2005, see Figure 4.4. Following Davison and Gholamrezaee [25], we fit a Schlather process with an isotropic powered exponential correlation function and

TABLE 4.4 – Distribution of the partition size for the temperature data estimated from a Markov chain of length 10000

Partition size	1	2	3	4	5–16
Empirical probabilities (%)	2·47	21·55	64·63	10·74	0·61

trend surfaces $\eta(t) = \beta_{0,\eta} + \beta_{1,\eta}\mathrm{alt}(t)$, $\sigma(t) = \beta_{0,\sigma}$, $\xi(t) = \beta_{0,\xi} + \beta_{1,\xi}\mathrm{alt}(t)$, where $\mathrm{alt}(t)$ denotes the altitude above mean sea level in kilometres and $\{\eta(t), \sigma(t), \xi(t)\}$ are the location, scale and shape parameters of the generalized extreme value distribution at location t. The spatial dependence parameter estimates are $\hat{\lambda} = 260\ (149)$ and $\hat{\kappa} = 0.52\ (0.12)$ and yield to a fitted extremal coefficient function similar to our test case ρ_3 in Section 4.4.

In year 2003, western Europe was hit by a severe heat wave believed to be the hottest one ever recorded since at most 1540 [7]. Switzerland was severely impacted by this event since the nationwide record temperature of 41·5°C was recorded that year in Grono, Graubunden, near Lugano. Consequently for our analysis we use as conditional event the maximum temperatures observed in summer 2003. Based on the fitted Schlather model, we simulate a Markov chain of effective length 10000 with a burn-in period of length 500 and a thinning lag of 100 iterations. The distribution of the partition size estimated from these Markov chains is shown in Table 4.4. Around 90% of the time the conditional realizations were a consequence of at most three extremal functions. Since our original observations were not summer maxima but maximum daily values, a close inspection of the times series in year 2003 reveals that the hottest temperatures occurred between the 11th and 13th of August and, to some extent, corroborates the distribution of Table 4.4.

The right panel of Figure 4.4 shows the spatial distribution of temperature anomalies, i.e., the difference between the pointwise conditional medians obtained from 10000 conditional simulations and the pointwise unconditional medians estimated from the fitted Schlather model. As expected, the largest deviations occur in the plateau region of Switzerland while appreciably smaller values are found in the Alps. The differences range between 2·5°C and 4·75°C and is consistent with the values reported by climatologists for mean values [7].

Acknowledgements

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Appendix

4.A Brown–Resnick model

For all $t \in \mathcal{X}^k$ and Borel set $A \subset \mathbb{R}^k$

$$\mu_t(A) = \int_0^\infty \mathrm{pr}\left[\zeta \exp\{W(t) - \gamma(t)\} \in A\right] \zeta^{-2} \mathrm{d}\zeta = \int_0^\infty \int_{\mathbb{R}^k} 1_{\{\zeta \exp\{y-\gamma(t)\} \in A\}} f_t(y) \mathrm{d}y \zeta^{-2} \mathrm{d}\zeta,$$

where f_t denotes the density of the random vector $W(t)$, i.e., a centered Gaussian random vector with covariance matrix Σ_t and variance $2\gamma(t)$. The change of variables $z = \zeta \exp\{y - \gamma(t)\}$ and $r = \log \zeta$ yields

$$\mu_t(A) = \int_{-\infty}^\infty \int_A f_t\{\log z - r + \gamma(t)\} \prod_{i=1}^k z_i^{-1} \mathrm{d}z e^{-r} \mathrm{d}r = \int_A \lambda_t(z) \mathrm{d}z$$

with

$$\lambda_t(z) = \prod_{i=1}^k z_i^{-1} \int_{-\infty}^\infty f_t\{\log z - r + \gamma(t)\} e^{-r} \mathrm{d}r.$$

Since

$$f_t\{\log z - r + \gamma(t)\} e^{-r} = (2\pi)^{-k/2} |\Sigma_t|^{-1/2} \exp\left\{-\frac{1}{2} P(r)\right\},$$

with

$$P(r) = r^2 1_k^T \Sigma_t^{-1} 1_k - 2r \left[1_k^T \Sigma_t^{-1} \{\log z + \gamma(t)\} - 1\right] + \{\log z + \gamma(t)\}^T \Sigma_t^{-1} \{\log z + \gamma(t)\},$$

standard computations for Gaussian integrals give

$$\lambda_t(z) = C_t \exp\left(-\frac{1}{2} \log z^T Q_t \log z + L_t \log z\right) \prod_{i=1}^k z_i^{-1}.$$

The conditional intensity function is

$$\begin{aligned} \lambda_{s|t,z}(u) \quad = \quad & \frac{C_{(s,t)}}{C_t} \exp\left\{-\frac{1}{2} \log{(u,z)}^T Q_{(s,t)} \log(u,z) + \right. \\ & \left. L_{(s,t)} \log(u,z) + \frac{1}{2} \log z^T Q_t \log z - L_t \log z \right\} \prod_{i=1}^m u_i^{-1}, \end{aligned}$$

and since $\log(u, z) = J_{m,k} \log u + \tilde{J}_{m,k} \log z$, it is not difficult to show that

$$\lambda_{s|t,z}(u) = \frac{C_{(s,t)}}{C_t} \exp\left\{-\frac{1}{2} (\log u - \mu_{s|t,z})^T \Sigma_{s|t}^{-1} (\log u - \mu_{s|t,z})\right\} \prod_{i=1}^m u_i^{-1}.$$

Finally, the relation $C_{(s,t)}/C_t = (2\pi)^{-m/2} |\Sigma_{s|t}|^{-1/2}$ is a simple consequence of the normalization $\int \lambda_{s|t,z}(u) \mathrm{d}u = 1$.

4.B Schlather model

For all $t \in \mathcal{X}^k$ and Borel sets $A \subset \mathbb{R}^k$

$$\mu_t(A) = \int_0^\infty \mathrm{pr}\left\{(2\pi)^{1/2}\zeta\varepsilon(t) \in A\right\}\zeta^{-2}\mathrm{d}\zeta = \int_0^\infty \int_{\mathbb{R}^k} 1_{\left\{(2\pi)^{1/2}\zeta y \in A\right\}} f_{\mathbf{t}}(y)\mathrm{d}y\zeta^{-2}\mathrm{d}\zeta,$$

where f_t denotes the density of the random vector $\varepsilon(t)$, i.e., a centered Gaussian random vector with covariance matrix Σ_t. The change of variable $z = (2\pi)^{1/2}\zeta y$ gives

$$\begin{aligned}
\Lambda_t(A) &= (2\pi)^{-k/2}\int_0^\infty \int_A f_t\left\{\frac{z}{(2\pi)^{1/2}\zeta}\right\}\zeta^{-(k+2)}\mathrm{d}z\mathrm{d}\zeta \\
&= (2\pi)^{-k}|\Sigma_t|^{-1/2}\int_0^\infty \int_A \exp\left(-\frac{1}{4\pi\zeta^2}z^T\Sigma_t^{-1}z\right)\zeta^{-(k+2)}\mathrm{d}z\mathrm{d}\zeta \\
&= (2\pi)^{-k}|\Sigma_t|^{-1/2}\int_A \frac{2\pi}{z^T\Sigma_t^{-1}z}\mathbb{E}[X^{k-1}]\mathrm{d}z, \qquad X \sim \text{Weibull}\left\{\left(\frac{4\pi}{z^T\Sigma_t^{-1}z}\right)^{1/2}, 2\right\} \\
&= (2\pi)^{-k}|\Sigma_t|^{-1/2}\int_A \frac{2\pi}{z^T\Sigma_t^{-1}z}\left(\frac{4\pi}{z^T\Sigma_t^{-1}z}\right)^{(k-1)/2}\Gamma\left(\frac{k+1}{2}\right)\mathrm{d}z \\
&= \int_A \lambda_{\mathbf{t}}(z)\mathrm{d}z,
\end{aligned}$$

where $\lambda_t(z) = \pi^{-(k-1)/2}|\Sigma_t|^{-1/2}a_t(z)^{-(k+1)/2}\Gamma\left\{(k+1)/2\right\}$ and $a_t(z) = z^T\Sigma_t^{-1}z$.

For all $u \in \mathbb{R}^m$ the conditional intensity function is

$$\lambda_{s|t,z}(u) = \pi^{-m/2}\frac{|\Sigma_{(s,t)}|^{-1/2}}{|\Sigma_t|^{-1/2}}\left\{\frac{a_{(s,t)}(u,z)}{a_t(z)}\right\}^{-(m+k+1)/2} a_t(z)^{-m/2}\frac{\Gamma\left(\frac{m+k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}.$$

We start by focusing on the ratio $a_{(s,t)}(u,z)/a_t(z)$. Since

$$\begin{aligned}
&\begin{bmatrix}\Sigma_s & \Sigma_{s:t}\\ \Sigma_{t:s} & \Sigma_t\end{bmatrix}^{-1} \\
= &\begin{bmatrix}(\Sigma_s - \Sigma_{s:t}\Sigma_t^{-1}\Sigma_{t:s})^{-1} & -(\Sigma_s - \Sigma_{s:t}\Sigma_t^{-1}\Sigma_{t:s})^{-1}\Sigma_{s:t}\Sigma_t^{-1}\\ -\Sigma_t^{-1}\Sigma_{t:s}(\Sigma_s - \Sigma_{s:t}\Sigma_t^{-1}\Sigma_{t:s})^{-1} & \Sigma_t^{-1} + \Sigma_t^{-1}\Sigma_{t:s}(\Sigma_s - \Sigma_{s:t}\Sigma_t^{-1}\Sigma_{t:s})^{-1}\Sigma_{s:t}\Sigma_t^{-1}\end{bmatrix},
\end{aligned}$$

straightforward algebra shows that

$$\frac{a_{(s,t)}(u,z)}{a_t(z)} = 1 + \frac{(u-\mu)^T\tilde{\Sigma}^{-1}(u-\mu)}{k+1}, \qquad \mu = \Sigma_{s:t}\Sigma_t^{-1}z, \quad \tilde{\Sigma} = \frac{a_t(z)}{k+1}\left(\Sigma_s - \Sigma_{s:t}\Sigma_t^{-1}\Sigma_{t:s}\right).$$

We now simplify the ratio $|\Sigma_{(s,t)}|/|\Sigma_t|$. Using the fact that

$$\Sigma_{(s,t)} = \begin{bmatrix}\Sigma_s & \Sigma_{s:t}\\ \Sigma_{t:s} & \Sigma_t\end{bmatrix} = \begin{bmatrix}\mathrm{I}_m & \Sigma_{s:t}\\ 0_{k,m} & \Sigma_t\end{bmatrix}\begin{bmatrix}\Sigma_s - \Sigma_{s:t}\Sigma_t^{-1}\Sigma_{t:s} & 0_{m,k}\\ \Sigma_t^{-1}\Sigma_{t:s} & \mathrm{I}_k\end{bmatrix},$$

combined with some more algebra yields

$$\frac{|\Sigma_{(s,t)}|}{|\Sigma_t|} = |\Sigma_s - \Sigma_{s:t}\Sigma_t^{-1}\Sigma_{t:s}| = \left\{\frac{k+1}{a_t(z)}\right\}^m |\tilde{\Sigma}|.$$

Using the two previous results it is easily found that

$$\lambda_{s|t,z}(u) = \pi^{-m/2}(k+1)^{-m/2}|\tilde{\Sigma}|^{-1/2}\left\{1 + \frac{(u-\mu)^T\tilde{\Sigma}^{-1}(u-\mu)}{k+1}\right\}^{-(m+k+1)/2} \frac{\Gamma\left(\frac{m+k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)},$$

which corresponds to the density of a multivariate Student distribution with the expected parameters.

Chapitre 5

Strong mixing properties of max-infinitely divisible random fields

Abstract

Let $\eta = (\eta(t))_{t\in T}$ be a sample continuous max-infinitely divisible random field on a locally compact metric space T. For a closed subset $S \subset T$, we denote by η_S the restriction of η to S. We consider $\beta(S_1, S_2)$, the absolute regularity coefficient between η_{S_1} and η_{S_2}, where S_1, S_2 are two disjoint closed subsets of T. Our main result is a simple upper bound for $\beta(S_1, S_2)$ involving the exponent measure μ of η : we prove that $\beta(S_1, S_2) \leq 2\int \mathbb{P}[\eta \not<_{S_1} f,\ \eta \not<_{S_2} f]\,\mu(df)$, where $f \not<_S g$ means that there exists $s \in S$ such that $f(s) \geq g(s)$.

If η is a simple max-stable random field, the upper bound is related to the so-called extremal coefficients : for countable disjoint sets S_1 and S_2, we obtain $\beta(S_1, S_2) \leq 4\sum_{(s_1,s_2)\in S_1\times S_2}(2 - \theta(s_1, s_2))$, where $\theta(s_1, s_2)$ is the pair extremal coefficient.

As an application, we show that these new estimates entail a central limit theorem for stationary max-infinitely divisible random fields on $\mathbb{Z}^d$. In the stationary max-stable case, we derive the asymptotic normality of three simple estimators of the pair extremal coefficient.

5.1 Introduction

Max-stable random fields turn out to be fundamental models for spatial extremes since they arise as the limit of rescaled maxima. More precisely, consider the component-wise maxima

$$\eta_n(t) = \max_{1\leq i\leq n} \xi_i(t), \quad t \in T,$$

of independent copies ξ_i, $i \geq 1$, of a random field $\xi = (\xi(t))_{t\in T}$. If the random field $\eta_n = (\eta_n(t))_{t\in T}$ converges in distribution, as $n \to \infty$, under suitable affine normalization, then its limit $\eta = \{\eta(t)\}_{t\in T}$ is necessarily max-stable. Therefore, max-stable random fields play a central role in extreme value theory, just like Gaussian random fields do in the classical statistical theory based on the central limit theorem.

Max-stable processes have been studied extensively in the last decades and many of their properties are well-understood. For example, the structure of their finite dimensional distributions is well known and insightful Poisson point process or spectral representations are available. Also the theory has been extended to max-infinitely divisible (max-i.d.) processes. See for example the seminal works by Resnick [62], de Haan [29, 30], de Haan and Pickands [33], Giné Hahn and Vatan [44], Resnick and Roy [63] and many others. More details and further references can be found in the monographs by Resnick [62] or de Haan and Fereira [31].

The questions of mixing and ergodicity of max-stable random processes indexed by $\mathbb{R}$ or $\mathbb{Z}$ have been addressed recently. First results by Weintraub [80] in the max-stable case have been completed by Stoev [73], providing necessary and sufficient conditions for mixing of max-stable processes based on their spectral representations. More recently, Kabluchko and Schlather [50] extend these results and obtain necessary and sufficient conditions for both mixing and ergodicity of max-i.d. random processes. They define the dependence function of a stationary max-i.d. random process $\eta = (\eta(t))_{t\in\mathbb{Z}}$ by

$$\tau_a(h) = \log \frac{\mathbb{P}[\eta(0) \leq a, \eta(h) \leq a]}{\mathbb{P}[\eta(0) \leq a]\mathbb{P}[\eta(h) \leq a]}, \quad a > \operatorname{essinf} \eta(0), \quad h \in \mathbb{Z}.$$

Then, it holds with $\ell = \operatorname{essinf} \eta(0)$:

- η is mixing if and only if for all $a > \ell$, $\tau_a(n) \to 0$ as $n \to +\infty$;
- η is ergodic if and only if for all $a > \ell$, $n^{-1} \sum_{h=1}^{n} \tau_a(h) \to 0$ as $n \to +\infty$.

Ergodicity is strongly connected to the strong law of large numbers via the ergodic theorem. The above results find natural applications in statistics to obtain strong consistency of several natural estimators based on non-independent but ergodic observations.

Going a step further, we address in this paper the issue of estimating the strong mixing coefficients of max-i.d. random fields. In some sense, ergodicity and mixing state that the restrictions η_{S_1} and η_{S_2} to two subsets S_1, S_2 become almost independent when the distance between S_1 and S_2 goes to infinity. Strong mixing coefficients make this statement quantitative : we introduce two standard mixing coefficients $\alpha(S_1, S_2)$ and $\beta(S_1, S_2)$ that measure how much η_{S_1} and η_{S_2} differ from independence. The rate of decay of those coefficients as the distance between S_1 and S_2 goes to infinity is a crucial point for the central limit theorem (see Appendix 5.A.3). As an application, we consider the asymptotic normality of three simple estimators of the extremal coefficients of a stationary max-stable random field on $\mathbb{Z}^d$ with standard unit Fréchet margins.

Our approach differs from those of Stoev [73] based on spectral representations and of Kabluchko and Schlather [50] based on exponent measures. It relies on the Poisson point process representation of max-i.d. random fields offered by Hahn, Giné and Vatan [44] (see also Appendix 5.A.1) and on the notions of extremal and subextremal points recently introduced by the authors [35]. Palm theory for Poisson point process and Campbell-Slyvniak formula are also a key tool (see Appendix 5.A.2).

The structure of the paper is the following : the framework and results are detailed in the next Section ; Section 3 is devoted to the proofs and an Appendix gathers some more technical details.

5.2 Framework and results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T be a locally compact metric space. We denote by $\mathbb{C}(T) = \mathbb{C}(T, [0, +\infty))$ the space of nonnegative continuous functions endowed with the topology of uniform convergence on compact sets and by $\mathcal{C}$ its Borel σ-field. A measure is said to be locally finite if it assigns finite measure to compact sets. Let μ be a locally finite Borel measure on $\mathbb{C}_0(T) = \mathbb{C}(T) \setminus \{0\}$ satisfying

$$\mu\big[\{f \in \mathbb{C}_0(T);\ \sup_K f > \varepsilon\}\big] < \infty \quad \text{for all compact } K \subset T \text{ and } \varepsilon > 0, \tag{5.1}$$

and Φ a Poisson point process on $\mathbb{C}_0(T)$ with intensity μ. More rigorously, we should consider Φ as a random point measure rather than as a random set of points, since there may be points with multiplicities. It is however standard to consider Φ as a random set of points with possible repetitions.

We consider the random process

$$\eta(t) = \max\{\phi(t),\ \phi \in \Phi\}, \quad t \in T,$$

with the convention that the maximum of the empty set is equal to 0. Condition (5.1) ensures that the random process η is continuous on T (see [44] and Appendix 5.A.1). Another property is worth noting : η is max-infinitely divisible. This means that for all $n \geq 1$, there exists independent and identically distributed continuous random fields $(\eta_{i,n})_{1 \leq i \leq n}$ such that

$$\eta \stackrel{\mathcal{L}}{=} \vee_{i=1}^n \eta_{i,n},$$

where $\vee$ stands for pointwise maximum and $\stackrel{\mathcal{L}}{=}$ for equality in distribution. Note that the max-infinite divisibility of η is a simple consequence of the superposition theorem for Poisson point processes. Furthermore, for all $t \in T$, the essential infimum of the random variable $\eta(t)$ is equal to 0. As shown by Giné, Hahn and Vatan [44], up to simple transformations, essentially all max-i.d. continuous random process on T can be obtained in this way (see Appendix 5.A.1). The measure μ is called the exponent measure associated to the max-i.d. process η. It should be stressed that Giné *et al* [44] deal with upper semi-continuous functions. For the sake of simplicity, we consider in this paper only continuous processes, even if the main results (Theorems 5.2.1 and 5.2.3) can be extended almost directly to cover the case of upper semi-continuous processes.

We now introduce the so-called α- and β-mixing coefficients. For more details on strong mixing conditions, the reader should refer to the recent survey by C.Bradley [12] or to the monographs [37, 65, 13, 14, 15, 34]. For $S \subset T$ a closed subset, we denote by $\mathcal{F}_S$ the σ-field generated by the random variables $\{\eta(s),\ s \in S\}$ and by $\mathcal{P}_S$ the distribution of the restriction η_S in the set $\mathbb{C}(S)$ of nonnegative continuous functions on S endowed with its Borel σ-field $\mathcal{C}_S$. Let $S_1, S_2 \subset T$ be disjoint closed subsets. The α-mixing coefficient introduced by Rosenblatt [66] between the σ-fields $\mathcal{F}_{S_1}$ and $\mathcal{F}_{S_2}$ is defined by

$$\alpha(S_1, S_2) = \sup\Big\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|;\ A \in \mathcal{F}_{S_1}, B \in \mathcal{F}_{S_2}\Big\}.$$

The β-mixing coefficient (or absolute regularity coefficient, see Volkonskii and Rozanov [77]) between the σ-fields $\mathcal{F}_{S_1}$ and $\mathcal{F}_{S_2}$ is given by

$$\beta(S_1,S_2) \;=\; \sup\Big\{|\mathcal{P}_{S_1\cup S_2}(C)-\mathcal{P}_{S_1}\otimes\mathcal{P}_{S_2}(C)|;\ C\in\mathcal{C}_{S_1\cup S_2}\Big\}. \tag{5.2}$$

Since S_1 and S_2 are disjoint closed subsets, $\mathbb{C}(S_1\cup S_2)$ is naturally identified with $\mathbb{C}(S_1)\times\mathbb{C}(S_2)$. We denote by $\|\cdot\|_{var}$ the total variation of a signed measure. Equivalent definitions of the β-mixing coefficient are

$$\begin{aligned}\beta(S_1,S_2) &= \|\mathcal{P}_{S_1\cup S_2}-\mathcal{P}_{S_1}\otimes\mathcal{P}_{S_2}\|_{var}\\ &= \frac{1}{2}\sup\Big\{\sum_{i=1}^{I}\sum_{j=1}^{J}|\mathbb{P}(A_i\cap B_j)-\mathbb{P}(A_i)\mathbb{P}(B_j)|\Big\}\end{aligned}$$

where the supremum is taken over all partitions $\{A_1,\ldots,A_I\}$ and $\{B_1,\ldots,B_J\}$ of Ω with the A_i's in $\mathcal{F}_{S_1}$ and the B_j's in $\mathcal{F}_{S_2}$. The following inequality is worth noting

$$\alpha(S_1,S_2)\le\frac{1}{2}\beta(S_1,S_2). \tag{5.3}$$

Our main result is the following.

Theorem 5.2.1. *Let η be a continuous max-i.d. process on T with exponent measure μ satisfying* (5.1). *Then, for all disjoint closed subsets $S_1,S_2\subset T$,*

$$\beta(S_1,S_2)\le 2\int_{\mathcal{C}_0}\mathbb{P}[f\not<_{S_1}\eta,\ f\not<_{S_2}\eta]\,\mu(df).$$

In the particular case when S_1 and S_2 are finite or countable (which naturally arise for example if $T=\mathbb{Z}^d$), we can provide an upper bound for the mixing coefficient $\beta(S_1,S_2)$ involving only the 2-dimensional marginal distributions of the process η.

For $(s_1,s_2)\in T^2$, let μ_{s_1,s_2} be the exponent measure of the max-i.d. random vector $(\eta(s_1),\eta(s_2))$ defined on $[0,+\infty)^2$ by

$$\mu_{s_1,s_2}(A)=\mu\big[\{f\in\mathbb{C}_0(T);\ (f(s_1),f(s_2))\in A\}\big],\quad A\subset[0,+\infty)^2 \text{ Borel set.}$$

Corollary 5.2.2. *If S_1 and S_2 are finite or countable disjoint closed subsets of T,*

$$\beta(S_1,S_2)\le 2\sum_{s_1\in S_1}\sum_{s_2\in S_2}\int\mathbb{P}[\eta(s_1)\le y_1,\ \eta(s_2)\le y_2]\,\mu_{s_1,s_2}(dy_1dy_2).$$

Next, we focus on simple max-stable random fields, where the phrase simple means that the marginals are standardized to the standard unit Fréchet distribution,

$$\mathbb{P}[\eta(t)\le y]=\exp[-y^{-1}]1_{\{y>0\}},\quad y\in\mathbb{R},\ t\in T.$$

In this framework, an insight into the dependence structure is given by the extremal coefficients $\theta(S)$, $S\subset T$ compact, defined by the relation

$$\mathbb{P}[\sup_{s\in S}\eta(s)\le y]=\exp[-\theta(S)y^{-1}],\quad y>0. \tag{5.4}$$

Theorem 5.2.3. *Let η be a continuous simple max-stable random field on T. For all compact $S \subset T$, the quantity*

$$C(S) = \mathbb{E}[\sup\{\eta(s)^{-1};\ s \in S\}] \tag{5.5}$$

is finite and furthermore :

- *For all disjoint compact subsets $S_1, S_2 \subset T$,*

$$\beta(S_1, S_2) \le 2\Big[C(S_1) + C(S_2)\Big]\Big[\theta(S_1) + \theta(S_2) - \theta(S_1 \cup S_2)\Big].$$

- *For $(S_{1,i})_{i \in I}$ and $(S_{2,j})_{j \in J}$ countable families of compact subsets of T such that $S_1 = \cup_{i \in I} S_{1,i}$ and $S_2 = \cup_{j \in J} S_{2,j}$ are disjoint,*

$$\beta(S_1, S_2) \le 2 \sum_{i \in I} \sum_{j \in J} \Big[C(S_{1,i}) + C(S_{2,j})\Big]\Big[\theta(S_{1,i}) + \theta(S_{2,j}) - \theta(S_{1,i} \cup S_{2,j})\Big].$$

In the particular case when S_1 and S_2 are finite or countable, the mixing coefficient $\beta(S_1, S_2)$ can be bounded from above in terms of the extremal coefficient function

$$\theta(s_1, s_2) = \theta(\{s_1, s_2\}), \quad s_1, s_2 \in T.$$

We recall the following basic properties : it always holds $\theta(s_1, s_2) \in [1, 2]$; $\theta(s_1, s_2) = 2$ iff $\eta(s_1)$ and $\eta(s_2)$ are independent ; $\theta(s_1, s_2) = 1$ iff $\eta(s_1) = \eta(s_2)$. Thus the extremal coefficient function gives some insight into the 2-dimensional dependence structure of the max-stable field η, although it does not characterize it completely.

Corollary 5.2.4. *Suppose η is a continuous simple max-stable random field on T. If S_1 and S_2 are finite or countable disjoint closed subsets of T, then*

$$\beta(S_1, S_2) \le 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [2 - \theta(s_1, s_2)].$$

Remark 5.2.5. It should be stressed that the the proof of Theorem 5.2.3 relies on the following inequality : if η is a max-stable process with exponent measure μ, then for all disjoint compact subsets $S_1, S_2 \subset T$

$$\int_{C_0} \mathbb{P}[f \not<_{S_1} \eta,\ f \not<_{S_2} \eta]\, \mu(df) \le \Big[C(S_1) + C(S_2)\Big]\Big[\theta(S_1) + \theta(S_2) - \theta(S_1 \cup S_2)\Big]$$

In view of Theorem 5.2.1, this inequality entail the first point of Theorem 5.2.3. When $S_1 = \{s_1\}$ and $S_2 = \{s_2\}$, we obtain

$$\int \mathbb{P}[\eta(s_1) \le y_1,\ \eta(s_2) \le y_2]\, \mu_{s_1, s_2}(dy_1 dy_2) \le 2[2 - \theta(s_1, s_2)].$$

This is used in the proof of Corollary 5.2.4.

As noted in the introduction, our main motivation for considering the strong mixing properties of max-i.d. random fields is to obtain central limit Theorems (CLTs) for stationary max-i.d. random fields. In this direction, we focus on stationary random fields on $T = \mathbb{Z}^d$ and our analysis relies on Bolthausen's CLT [10] (see Appendix 5.A.3).

We denote by $|h| = \max_{1\le i\le d} |h_i|$ the norm of $h \in \mathbb{Z}^d$ and by $|S|$ the number of elements of a subset $S \subset \mathbb{Z}^d$. The boundary ∂S of S is the set of elements $h \in S$ such that there is $h' \notin S$ with $d(h, h') = 1$.

A random field $X = (X(t))_{t\in\mathbb{Z}^d}$ is said to be stationary if the law of $(X(t+s))_{t\in\mathbb{Z}^d}$ does not depend on $s \in \mathbb{Z}^d$. We say that a square integrable stationary random field X satisfies the CLT if the following two conditions are satisfied :

i) the series $\sigma^2 = \sum_{t\in\mathbb{Z}^d} \mathrm{Cov}[X(0), X(t)]$ converges absolutely ;

ii) for all sequence Λ_n of finite subsets of $\mathbb{Z}^d$, which increases to $\mathbb{Z}^d$ and such that we have $\lim_{n\to\infty} |\partial\Lambda_n|/|\Lambda_n| = 0$, the sequence $|\Lambda_n|^{-1/2} \sum_{t\in\Lambda_n}(X(t) - \mathbb{E}[X(t)])$ converges in law to the normal distribution with mean 0 and variance σ^2 as $n \to \infty$.

Please note we do not require the limit variance σ^2 to be positive ; the case $\sigma^2 = 0$ corresponds to a degenerated CLT where the limit distribution is the Dirac mass at zero. Bolthausen's CLT for stationary mixing random fields together with our estimates of mixing coefficients of max-i.d. random fields yield the following Theorem.

Theorem 5.2.6. *Suppose η is a stationary max-i.d. random field on $\mathbb{Z}^d$ with exponent measure μ and let*

$$\gamma(h) = \int \mathbb{P}[\eta(0) \le y_1,\ \eta(h) \le y_2]\, \mu_{0,h}(dy_1 dy_2), \quad h \in \mathbb{Z}^d.$$

Let $g : \mathbb{R}^p \to \mathbb{R}$ be a measurable function and $t_1, \ldots, t_p \in \mathbb{Z}^d$ such that

$$\mathbb{E}[g(\eta(t_1), \ldots, \eta(t_p))^{2+\delta}] < \infty \ \textit{for some}\ \ \delta > 0,$$

and assume that

$$\sum_{|h|\ge m} \gamma(h) = o(m^{-d}) \quad \textit{and} \quad \sum_{m=1}^{\infty} m^{d-1} \sup_{|h|\ge m} \gamma(h)^{\delta/(2+\delta)} < \infty. \tag{5.6}$$

Then the stationary random field X defined by

$$X(t) = g(\eta(t_1 + t), \ldots, \eta(t_p + t)), \quad t \in \mathbb{Z}^d$$

satisfies the CLT.

Condition (5.6) requires that γ goes fast enough to 0 at infinity. It is met for example if

$$\gamma(h) \le C \cdot |h|^{-b} \quad \text{for some} \quad b > d\max(2, (2+\delta)/\delta) \ \text{and} \ C > 0. \tag{5.7}$$

If η is simple max-stable, it follows from the proof of Theorem 5.2.3 and Corollary 5.2.4 that

$$\gamma(h) \le 2(2 - \theta(0, h)),$$

with θ the extremal coefficient function.

As an application of Theorem 5.2.6, we consider the estimation of the extremal coefficient for a stationary simple max-stable random field on $\mathbb{Z}^d$. For $h \in \mathbb{Z}^d$, we note $\theta(h) = \theta(0,h)$. Equation (5.4) implies

$$\theta(h) = -y \log p(h,y) \quad \text{with} \quad p(h,y) = \mathbb{P}(\eta(0) \leq y,\ \eta(h) \leq y), \quad y > 0,$$

suggesting the simple estimator

$$\hat{\theta}_n^{(1)}(h) = -y \log \hat{p}_n(h,y) \quad \text{with} \quad \hat{p}_n(h,y) = |\Lambda_n|^{-1} \sum_{t \in \Lambda_n} 1_{\{\eta(t) \leq y,\ \eta(t+h) \leq y\}} \tag{5.8}$$

where Λ_n is a sequence of finite subsets increasing to $\mathbb{Z}^d$ such that $|\partial \Lambda_n|/|\Lambda_n| \to 0$ as $n \to \infty$. The fact that the naive estimator $\hat{\theta}_n^{(1)}(h)$ depends of the threshold level $y > 0$ is not satisfactory. Alternatively, one may consider the following procedures. Smith [70] noticed that $\min(\eta(0)^{-1}, \eta(h)^{-1})$ has an exponential distribution with mean $\theta(h)^{-1}$ and proposed the estimator

$$\hat{\theta}_n^{(2)}(h) = \frac{|\Lambda_n|}{\sum_{t \in \Lambda_n} \min(\eta(t)^{-1}, \eta(t+h)^{-1})}.$$

Cooley et al. [19] introduced the F-madogram defined by

$$\nu_F(h) = \mathbb{E}[|F(\eta(0)) - F(\eta(h))|] \quad \text{with} \quad F(y) = \exp(-1/y) 1_{\{y>0\}}$$

and showed that it satisfies

$$\nu_F(h) = \frac{1}{2} \frac{\theta(h) - 1}{\theta(h) + 1} \quad \text{or equivalently} \quad \theta(h) = \frac{1 + 2\nu_F(h)}{1 - 2\nu_F(h)}.$$

This suggests the estimator

$$\hat{\theta}_n^{(3)}(h) = \frac{|\Lambda_n| + 2\sum_{t \in \Lambda_n} |F(\eta(t)) - F(\eta(t+h))|}{|\Lambda_n| - 2\sum_{t \in \Lambda_n} |F(\eta(t)) - F(\eta(t+h))|}.$$

The following Proposition states the asymptotic normality of these estimators.

Proposition 5.2.7. *Suppose that η is a stationary simple max-stable random field on $\mathbb{Z}^d$ with extremal coefficient function satisfying*

$$2 - \theta(h) \leq C \cdot |h|^{-b} \quad \textit{for some} \quad b > 2d \ \textit{and} \ C > 0. \tag{5.9}$$

Then, the estimators $\hat{\theta}_n^{(i)}(h)$, $i = 1, 2, 3$ are asymptotically normal :

$$|\Lambda_n|^{1/2} \big(\hat{\theta}_n^{(i)}(h) - \theta(h) \big) \implies \mathcal{N}(0, \sigma_i^2) \quad \textit{as } n \to \infty$$

with limit variances

$$\begin{aligned}
\sigma_1^2 &= y^2 \sum_{t\in\mathbb{Z}^d} \big(\exp[(2\theta(h)-\theta(\{0,h,t,t+h\}))y^{-1}]-1\big),\\
\sigma_2^2 &= \theta(h)^4 \sum_{t\in\mathbb{Z}^d} \mathrm{Cov}\big[\min(\eta(0)^{-1},\eta(h)^{-1})\,,\,\min(\eta(t)^{-1},\eta(t+h)^{-1})\big],\\
\sigma_3^2 &= (\theta(h)+1)^4 \sum_{t\in\mathbb{Z}^d} \mathrm{Cov}\big[|F(\eta(0))-F(\eta(h))|\,,\,|F(\eta(t))-F(\eta(t+h))|\big].
\end{aligned}$$

Interestingly, the function $y \mapsto \sigma_1^2$ is strictly convex, has limit $+\infty$ as $y \to 0^+$ or $+\infty$ and hence it admits a unique minimizer $y^\star$ corresponding to an asymptotically optimal truncation level for the estimator $\hat\theta_n^{(1)}$. Unfortunately, the limit variances σ_2^2 and σ_3^2 are not so explicit so that a comparison between the three is difficult.

We illustrate our results on two classes of stationary max-stable random fields on $\mathbb{R}^d$.

Example 5.2.1. We consider the Brown-Resnick simple max-stable model (see Kabluchko et al. [51]). Let $(W_i)_{i\geq 1}$ be independent copies of a sample continuous stationary increments Gaussian random field $W = (W(t))_{t\in\mathbb{R}^d}$ with zero mean and variance $\sigma^2(t)$. Independently, let $(Z_i)_{i\geq 1}$ be the nonincreasing enumeration of the points of a Poisson point process $(0,+\infty)$ with intensity $z^{-2}dz$. The associated Brown-Resnick max-stable random field is defined by

$$\eta(t) = \bigvee_{i=1}^{\infty} Z_i \exp[W_i(t) - \sigma^2(t)/2], \quad t \in \mathbb{R}^d.$$

It is known that η is a stationary simple max-stable random field whose law depends only on the negative semi-definite function V, called the variogram of W, and defined by

$$V(h) = \mathbb{E}[(W(t+h)-W(t))^2], \quad h \in \mathbb{R}^d.$$

In this case, the extremal coefficient function is given by

$$\theta(s_1,s_2) = 2\Psi(\sqrt{V(s_2-s_1)}/2), \quad s_1,s_2 \in \mathbb{R}^d,$$

where Ψ denotes the cdf of the standard normal law. Using the tail equivalent

$$1-\Psi(x) \sim \frac{e^{-x^2/2}}{x\sqrt{2\pi}} \quad \text{as } x \to +\infty,$$

we see that Equation (5.9) holds as soon as

$$\liminf_{h\to\infty} \frac{V(h)}{\sqrt{\log|h|}} > 2\sqrt{d}.$$

This completes the necessary and sufficient conditions for ergodicity or mixing of Brown-Resnick processes given by Kabluchko and Schlather [50].

Example 5.2.2. Our second class of example is the moving maximum process by de Haan and Pereira [32]. Let $f : \mathbb{R}^d \to [0, +\infty)$ be a continuous density function such that

$$\int_{\mathbb{R}^d} f(x)\, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} \sup_{|h|\leq 1} f(x+h)\, dx < \infty.$$

Let $\sum_{i=1}^{\infty} \delta_{(Z_i,U_i)}$ be a Poisson random measure on $(0, +\infty) \times \mathbb{R}^d$ with intensity $z^{-2}dzdu$. Then the random field

$$\eta(t) = \bigvee_{i=1}^{\infty} Z_i f(t - U_i), \quad t \in \mathbb{R}^d,$$

is a stationary sample continuous simple max-stable random field. The corresponding extremal coefficient function is given by

$$\theta(s_1, s_2) = \int_{\mathbb{R}^d} \max(f(s_1 - x), f(s_2 - x))\, dx, \quad s_1, s_2 \in \mathbb{R}^d.$$

Some computations reveals that Equation (5.9) holds true as soon as

$$\limsup_{h\to\infty} \frac{\log f(h)}{\log |h|} < -\kappa_d$$

with $\kappa_1 = 3$ and $\kappa_d = 2(d + 1)$ for $d \geq 2$.

5.3 Proofs

5.3.1 Strong mixing properties of extremal point processes

In the sequel, we shall write shortly $\mathbb{C}_0 = \mathbb{C}_0(T)$. We denote by $M_p(\mathbb{C}_0)$ the set of locally finite point measures N on $\mathbb{C}_0$ endowed with the σ-algebra generated by the family of mappings $\{N \mapsto N(A),\ A \subset \mathbb{C}_0 \text{ Borel set}\}$. We introduce here the notion of S-extremal points that will play a key role in this work. We use the following notations : if f_1, f_2 are two functions defined (at least) on S, we note

$$\begin{aligned} f_1 =_S f_2 \quad &\text{if and only if} \quad \forall s \in S,\ f_1(s) = f_2(s), \\ f_1 <_S f_2 \quad &\text{if and only if} \quad \forall s \in S,\ f_1(s) < f_2(s), \\ f_1 \not<_S f_2 \quad &\text{if and only if} \quad \exists s \in S,\ f_1(s) \geq f_2(s). \end{aligned}$$

A point $\phi \in \Phi$ is said to be S-subextremal if $\phi <_S \eta$, it is said S-extremal otherwise, i.e. if there exists $s \in S$ such that $\phi(s) = \eta(s)$. In words, a S-subextremal point has no contribution to the maximum η on S.

Definition 5.3.1. *Define the S-extremal random point process Φ_S^+ and the S-subextremal random point process Φ_S^- by*

$$\Phi_S^+ = \{\phi \in \Phi;\ \phi \not<_S \eta\} \quad \textit{and} \quad \Phi_S^- = \{\phi \in \Phi;\ \phi <_S \eta\}.$$

The fact that Φ_S^+ and Φ_S^- are well defined point processes, i.e. that they satisfy some measurability properties, is proved in [35] Appendix A3. Clearly, the restriction η_S depends on Φ_S^+ only :

$$\eta(s) = \max\{\phi(s);\ \phi \in \Phi_S^+\}, \quad s \in S.$$

This implies that the strong mixing coefficient $\beta(S_1, S_2)$ defined by Equation (5.2) can be upper bounded by a similar β-mixing coefficient defined on the level of the extremal point process $\Phi_{S_1}^+$, $\Phi_{S_2}^+$. For $i = 1, 2$, let $P_{\Phi_{S_i}^+}$ the distribution of $\Phi_{S_i}^+$ on the space of locally finite point measures on $\mathbb{C}_0$ and let $P_{(\Phi_{S_1}^+, \Phi_{S_2}^+)}$ be the joint distribution of $(\Phi_{S_1}^+, \Phi_{S_2}^+)$. We define

$$\beta(\Phi_{S_1}^+, \Phi_{S_2}^+) = \| P_{(\Phi_{S_1}^+, \Phi_{S_2}^+)} - P_{\Phi_{S_1}^+} \otimes P_{\Phi_{S_2}^+} \|_{var}. \tag{5.10}$$

It holds

$$\beta(S_1, S_2) \leq \beta(\Phi_{S_1}^+, \Phi_{S_2}^+). \tag{5.11}$$

The following Theorem provides a simple estimate for the β-mixing coefficient on the point process level. It implies Theorem 5.2.1 straightforwardly and has a clearer interpretation.

Theorem 5.3.2. – *The following upper bound holds true :*

$$\beta(\Phi_{S_1}^+, \Phi_{S_2}^+) \leq 2\, \mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset] \tag{5.12}$$

with

$$\mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset] \leq \int_{\mathcal{C}_0} \mathbb{P}(f \not<_{S_1} \eta,\ f \not<_{S_2} \eta)\, \mu(df). \tag{5.13}$$

– *If the point process Φ is simple (in particular in the max-stable case), the following lower bound holds true :*

$$\beta(\Phi_{S_1}^+, \Phi_{S_2}^+) \geq \mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset] \tag{5.14}$$

Clearly, equations (5.11), (5.12) and (5.13) together imply Theorem 5.2.1.

Remark 5.3.3. The upper and lower bound in Theorem 5.3.2 are of the same order, and hence relatively sharp. It is not clear however how to bound $\beta(S_1, S_2)$ from below and how sharp the upper bound in Theorem 5.2.1 is.

5.3.2 Proof of Theorem 5.3.2

The upper bound for the mixing coefficient $\beta(\Phi_{S_1}^+, \Phi_{S_2}^+)$ defined by Equation (5.10) relies on a standard coupling argument. There are indeed deep relationships between β-mixing and optimal couplings and we will use the following result (see e.g. [65] chapter 5). Note that the lemma holds true for any pairs of random variables, but for the sake of future reference, we state it for extremal point processes.

Lemma 5.3.4. *On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that the random variables $(\Phi_{S_1}^{+i}, \Phi_{S_2}^{+i})$, $i = 1, 2$ are such that :*

i) the distribution of $(\Phi_{S_1}^{+1}, \Phi_{S_2}^{+1})$ *is* $P_{(\Phi_{S_1}^+,\Phi_{S_2}^+)}$ *;*

ii) the distribution of $(\Phi_{S_1}^{+2}, \Phi_{S_2}^{+2})$ *is* $P_{\Phi_{S_1}^+} \otimes P_{\Phi_{S_2}^+}$.

Then,

$$\beta(\Phi_{S_1}^+, \Phi_{S_2}^+) \le \mathbb{P}\Big[(\Phi_{S_1}^{+1}, \Phi_{S_2}^{+1}) \neq (\Phi_{S_1}^{+2}, \Phi_{S_2}^{+2})\Big]$$

We say that the random variables $(\Phi_{S_1}^{+i}, \Phi_{S_2}^{+i})$, $i = 1, 2$ satisfying i) and ii) realize a coupling between the distributions $P_{(\Phi_{S_1}^+,\Phi_{S_2}^+)}$ and $P_{\Phi_{S_1}^+} \otimes P_{\Phi_{S_2}^+}$.

In order to construct a suitable coupling, we need the following Lemma describing the dependence between Φ_S^+ and Φ_S^-.

Lemma 5.3.5. *Let $S \subset T$ be a closed set. The conditional distribution of Φ_S^- with respect to Φ_S^+ is equal to the distribution of a Poisson point process with intensity $1_{\{f <_S \eta\}}\mu(df)$.*

Proof of Lemma 5.3.5:
Note that in the particular case when T is compact and Φ_S^+ is finite almost surely, Lemma 5.3.5 follows from [35] Theorem 2.1 and Corollary 2.1. For T non compact, the proof need to be modified in a non straightforward way.

Clearly, the event $\{f <_S \eta\}$ depends only on the restriction η_S and is hence measurable with respect to the σ-field generated by Φ_S^+. In order to prove the statement, let $A_1, \dots, A_k \subset \mathbb{C}_0$ be disjoint compact sets and $n_1, \dots, n_k \ge 0$. Let $A = \cup_{i=1}^k A_i$ and $n = \sum_{i=1}^k n_i$. We compute the conditional probability with respect to Φ_S^+ of the event

$$\{\Phi_S^-(A_1) = n_1, \dots, \Phi_S^-(A_k) = n_k\}.$$

This event is equal to $\{\Phi_S^- \in B\}$ with $B = \{N \in M_p(\mathbb{C}_0);\ N(A_1) = n_1, \dots, N(A_k) = n_k\}$. We remark that it is realized if and only if there exists a n-uplet $(\phi_1, \dots, \phi_n)$ of atoms of Φ such that :

- the atoms $\phi_1, \dots, \phi_n$ are S-subextremal ;
- $\sum_{j=1}^n \delta_{\phi_j} \in B$;
- the point measure $\Phi - \sum_{j=1}^n \delta_{\phi_j}$ has no S-subextremal atom in A, i.e. it belongs to

$$D = \{N \in M_p(\mathbb{C}_0);\ N_S^-(A) = 0\}.$$

Then the n-uplet $(\phi_1, \dots, \phi_n)$ is unique up to permutation of the coordinates. The above observations entail that for all measurable $C \subset M_p(\mathbb{C}_0)$,

$$\begin{aligned}\mathbb{P}[\Phi_S^+ \in C,\ \Phi_S^- \in B] \ &= \ \frac{1}{n!}\mathbb{E}\Big[\int_{\mathcal{C}_0^n} 1_{\{\Phi_S^+ \in C\}} 1_{\{\forall i \in [\![1,n]\!],\ \phi_i <_S \eta\}} 1_{\{\sum_{i=1}^n \delta_{\phi_i} \in B\}} 1_{\{\Phi - \sum_{i=1}^n \delta_{\phi_i} \in D\}} \\ &\qquad \Phi(d\phi_1)(\Phi - \delta_{\phi_1})(d\phi_2) \cdots (\Phi - \sum_{i=1}^{n-1} \delta_{\phi_i})(d\phi_n)\Big].\end{aligned}$$

Campbell-Slyvniak formula entails

$$\begin{aligned}&\mathbb{P}[\Phi_S^+ \in C,\ \Phi_S^- \in B] \\ = \ &\mathbb{E}\Big[1_{\{\Phi_S^+ \in C\}} 1_{\{\Phi_S^-(A) = 0\}} \frac{1}{n!} \int_{\mathcal{C}_0^n} 1_{\{\sum_{i=1}^n \delta_{f_i} \in B\}} \otimes_{i=1}^n \left(1_{\{f_i <_S \eta\}} \mu(df_i)\right)\Big]. \qquad (5.15)\end{aligned}$$

Summing this relation over the different values of $n_1, \dots, n_k \in \mathbb{N}$ and the related sets $B = \{N \in M_p(\mathbb{C}_0);\ N(A_1) = n_1, \dots, N(A_k) = n_k\}$, we obtain

$$\mathbb{P}[\Phi_S^+ \in C] = \mathbb{E}\Big[1_{\{\Phi_S^+ \in C\}} 1_{\{\Phi_S^-(A)=0\}} \exp\big[\mu(\{f \in A;\ f <_S \eta\})\big]\Big].$$

So we can rewrite Equation (5.15) as

$$\mathbb{P}[\Phi_S^+ \in C,\ \Phi_S^- \in B] \;=\; \mathbb{E}\Big[1_{\{\Phi_S^+ \in C\}} 1_{\{\Phi_S^-(A)=0\}} \exp\big[\mu(\{f \in A;\ f <_S \eta\})\big] K(\eta_S, B)\Big],$$

where

$$K(\eta_S, B) = \frac{\exp\big[-\mu(\{f \in A;\ f <_S \eta\})\big]}{n!} \int_{A^n} 1_{\{\sum_{i=1}^n \delta_{f_i} \in B\}} \otimes_{i=1}^n \big(1_{\{f_i <_S \eta\}} \mu(df_i)\big)$$

is the conditional probability of $\{\Phi_S^- \in B\}$ with respect to Φ_S^+ (note it depends on Φ_S^+ only through the restriction η_S). We recognize the distribution of a Poisson random measure with intensity $1_{\{f <_S \eta\}} \mu(df)$ and this proves Lemma 5.3.5. □

We now construct the coupling providing the upper bound (5.12).

Proposition 5.3.6. *Let $(\widetilde{\Phi}, \widetilde{\eta})$ be an independent copy of (Φ, η) and define*

$$\widehat{\Phi} = \Phi_{S_1}^+ \cup \{\widetilde{\phi} \in \widetilde{\Phi};\ \widetilde{\phi} <_{S_1} \eta\}. \tag{5.16}$$

The following properties hold true :

- *$\widehat{\Phi}$ has the same distribution as Φ and $\widetilde{\Phi}$ and satisfies*

$$\widehat{\Phi}_{S_1}^+ = \Phi_{S_1}^+ \quad , \quad \widehat{\Phi}_{S_1}^- = \{\widetilde{\phi} \in \widetilde{\Phi};\ \widetilde{\phi} <_{S_1} \eta\}; \tag{5.17}$$

- *$(\widehat{\Phi}_{S_1}^+, \widehat{\Phi}_{S_2}^+)$ and $(\Phi_{S_1}^+, \widetilde{\Phi}_{S_2}^+)$ is a coupling between $P_{(\Phi_{S_1}^+, \Phi_{S_2}^+)}$ and $P_{\Phi_{S_1}^+} \otimes P_{\Phi_{S_2}^+}$ such that*

$$\mathbb{P}\Big[(\widehat{\Phi}_{S_1}^+, \widehat{\Phi}_{S_2}^+) \neq (\Phi_{S_1}^+, \widetilde{\Phi}_{S_2}^+)\Big] \leq 2\, \mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset]. \tag{5.18}$$

Proof of Proposition 5.3.6:

- Equation (5.17) follows from the construction of $\widehat{\Phi}$: consider

  $$\widehat{\eta}(t) = \bigvee_{\phi \in \widehat{\Phi}} \phi(t), \quad t \in T;$$

  the maximum η is achieved on S_1 by the S_1-extremal points $\Phi_{S_1}^+$, and the definition (5.16) ensures that $\widehat{\eta}$ and η are equal on S_1 so that equation (5.17) holds.
  Furthermore, conditionally on $\Phi_{S_1}^+$, the distribution of $\{\widetilde{\phi} \in \widetilde{\Phi};\ \widetilde{\phi} <_{S_1} \eta\}$ is equal to the distribution of a Poisson point process with intensity $1_{\{f <_{S_1} \eta\}} \mu(df)$. According to Lemma 5.3.5, this is the conditional distribution of $\Phi_{S_1}^-$ given $\Phi_{S_1}^+$, whence $(\widehat{\Phi}_{S_1}^+, \widehat{\Phi}_{S_1}^-)$ has the same distribution as $(\Phi_{S_1}^+, \Phi_{S_1}^-)$. We deduce that $\Phi = \Phi_{S_1}^+ \cup \Phi_{S_1}^-$ and $\widehat{\Phi} = \widehat{\Phi}_{S_1}^+ \cup \widehat{\Phi}_{S_1}^-$ have the same distribution.

– The coupling property is easily proved : since Φ and $\widehat{\Phi}$ have the same distribution, the law of $(\widehat{\Phi}^+_{S_1}, \widehat{\Phi}^+_{S_2})$ is equal to $P_{(\Phi^+_{S_1}, \Phi^+_{S_2})}$; since Φ and $\widetilde{\Phi}$ are independent, $(\Phi^+_{S_1}, \widetilde{\Phi}^+_{S_2})$ has law $P_{\Phi^+_{S_1}} \otimes P_{\Phi^+_{S_2}}$.
We are left to prove Equation (5.18). Since $\Phi^+_{S_1} = \widehat{\Phi}^+_{S_1}$, we need to bound the probability $\mathbb{P}\big[\widehat{\Phi}^+_{S_2} \neq \widetilde{\Phi}^+_{S_2}\big]$ from above. By construction, $\widehat{\Phi}$ is obtained from $\widetilde{\Phi}$ by removing the points $\widetilde{\phi} \in \widetilde{\Phi}$ such that $\widetilde{\phi} \not<_{S_1} \eta$ and adding the points $\phi \in \Phi^+_{S_1}$. Hence, it holds

$$\{\widehat{\Phi}^+_{S_2} \neq \widetilde{\Phi}^+_{S_2}\} \ \subset \ \{\exists \phi \in \Phi^+_{S_1},\ \phi \not<_{S_2} \widehat{\eta}\} \ \cup \ \{\exists \widetilde{\phi} \in \widetilde{\Phi}^+_{S_2},\ \widetilde{\phi} \not<_{S_1} \eta\}$$

Noting the equality of events

$$\{\exists \phi \in \Phi^+_{S_1},\ \phi \not<_{S_2} \widehat{\eta}\} = \{\exists \phi \in \widehat{\Phi}^+_{S_1} \cap \widehat{\Phi}^+_{S_2}\} = \{\widehat{\Phi}^+_{S_1} \cap \widehat{\Phi}^+_{S_2} \neq \emptyset\},$$

we obtain

$$\mathbb{P}[\widehat{\Phi}^+_{S_2} \neq \widetilde{\Phi}^+_{S_2}] \leq \mathbb{P}[\widehat{\Phi}^+_{S_1} \cap \widehat{\Phi}^+_{S_2} \neq \emptyset] + \mathbb{P}[\exists \widetilde{\phi} \in \widetilde{\Phi}^+_{S_2},\ \widetilde{\phi} \not<_{S_1} \eta].$$

Since $\widehat{\Phi}$ and Φ have the same law, we have

$$\mathbb{P}[\widehat{\Phi}^+_{S_1} \cap \widehat{\Phi}^+_{S_2} \neq \emptyset] = \mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset].$$

Hence, equation (5.18) follows from the upper bound

$$\mathbb{P}[\exists \widetilde{\phi} \in \widetilde{\Phi}^+_{S_2},\ \widetilde{\phi} \not<_{S_1} \eta] \leq \mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset]$$

that we prove now. Using symmetry and exchanging the roles of Φ and $\widetilde{\Phi}$ on the one hand and the roles of S_1 and S_2 on the other hand, it is equivalent to prove that

$$\mathbb{P}[\exists \phi \in \Phi^+_{S_1},\ \phi \not<_{S_2} \widetilde{\eta}] \leq \mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset].$$

We conclude the proof by noticing that the inclusion of events

$$\{\exists \phi \in \Phi^+_{S_1},\ \phi \not<_{S_2} \widetilde{\eta}\} \subset \{\exists \phi \in \Phi^+_{S_1},\ \phi \not<_{S_2} \widehat{\eta}\}$$

entails

$$\mathbb{P}[\exists \phi \in \Phi^+_{S_1},\ \phi \not<_{S_2} \widetilde{\eta}] \leq \mathbb{P}[\exists \phi \in \Phi^+_{S_1},\ \phi \not<_{S_2} \widehat{\eta}] = \mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset].$$

□

We now complete the proof of Theorem 5.3.2 by proving Equations (5.13) and (5.14).
Proof of Equation (5.13)*:*
We observe that

$$\{\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset\} = \{\exists \phi \in \Phi,\ \phi \not<_{S_1} \eta,\ \phi \not<_{S_2} \eta\}$$

which entails

$$\mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset] \leq \mathbb{E}\Big[\sum_{\phi\in\Phi} 1_{\{\phi\not<_{S_1}\eta,\ \phi\not<_{S_2}\eta\}}\Big].$$

Noting that $\phi \not<_{S_i} \eta$ if and only if $\phi \not<_{S_i} \max(\Phi - \{\phi\})$, we apply Campbell-Slyvniak formula (see Appendix 5.A.2) and compute

$$\begin{aligned}
\mathbb{E}\Big[\sum_{\phi\in\Phi} 1_{\{\phi\not<_{S_1}\eta,\ \phi\not<_{S_2}\eta\}}\Big] &= \mathbb{E}\Big[\sum_{\phi\in\Phi} 1_{\{\phi\not<_{S_1}\max(\Phi-\{\phi\}),\ \phi\not<_{S_2}\max(\Phi-\{\phi\})\}}\Big] \\
&= \int_{\mathcal{C}_0} \mathbb{E}[1_{\{f\not<_{S_1}\max(\Phi),\ f\not<_{S_2}\max(\Phi)\}}]\mu(df) \\
&= \int_{\mathcal{C}_0} \mathbb{P}[f \not<_{S_1} \eta,\ f \not<_{S_2} \eta]\mu(df).
\end{aligned}$$

□

Proof of Equation (5.14)*:*
For any measurable subset $C \subset M_p(\mathbb{C}_0) \times M_p(\mathbb{C}_0)$,

$$\beta(\Phi_{S_1}^+, \Phi_{S_2}^+) \geq |\mathbb{P}[(\Phi_{S_1}^+, \Phi_{S_2}^+) \in C] - \mathbb{P}[(\Phi_{S_1}^+, \widetilde{\Phi}_{S_2}^+) \in C]|$$

where $\widetilde{\Phi}$ is an independent copy of Φ. We obtain the lower bound (5.14) by choosing the subset

$$C = \{(M_1, M_2) \in M_p(\mathbb{C}_0) \times M_p(\mathbb{C}_0);\ M_1 \cap M_2 \neq \emptyset\}.$$

This yields indeed

$$\beta(\Phi_{S_1}^+, \Phi_{S_2}^+) \geq |\mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset] - \mathbb{P}[\Phi_{S_1}^+ \cap \widetilde{\Phi}_{S_2}^+ \neq \emptyset]|.$$

and in the case when Φ is a simple point process, i.e. when the intensity measure μ has no atom, we have

$$\mathbb{P}[\Phi_{S_1}^+ \cap \widetilde{\Phi}_{S_2}^+ \neq \emptyset] = \mathbb{P}[\Phi \cap \widetilde{\Phi} \neq \emptyset] = 0.$$

□

5.3.3 Proof of Corollaries 5.2.2 and 5.2.4 and Theorem 5.2.3

Proof of Corollary 5.2.2:
We have for all $f \in \mathbb{C}_0$,

$$\begin{aligned}
\{f \not<_{S_1} \eta,\ f \not<_{S_2} \eta\} &= \{\exists (s_1, s_2) \in S_1 \times S_2,\ f(s_1) \geq \eta(s_1),\ f(s_2) \geq \eta(s_2)\} \\
&= \cup_{s_1\in S_1} \cup_{s_2\in S_2} \{\eta(s_1) \leq f(s_1),\ \eta(s_2) \leq f(s_2)\}
\end{aligned}$$

whence, for S_1 and S_2 finite or countable,

$$\mathbb{P}[f \not<_{S_1} \eta,\ f \not<_{S_2} \eta] \leq \sum_{s_1\in S_1}\sum_{s_2\in S_2} \mathbb{P}[\eta(s_1) \leq f(s_1),\ \eta(s_2) \leq f(s_2)].$$

As a consequence, the integral in Theorem 5.2.1 satisfies

$$\begin{aligned}
&\int_{\mathcal{C}_0} \mathbb{P}[f \not<_{S_1} \eta,\ f \not<_{S_2} \eta]\, \mu(df) \\
\leq\ & \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \int_{\mathcal{C}_0} \mathbb{P}[\eta(s_1) \leq f(s_1),\ \eta(s_2) \leq f(s_2)]\, \mu(df) \\
=\ & \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \int_{[0,+\infty)^2} \mathbb{P}[\eta(s_1) \leq y_1,\ \eta(s_2) \leq y_2]\, \mu_{s_1,s_2}(dy_1 dy_2).
\end{aligned}$$

In the last line, we have used the fact that μ_{s_1,s_2} is the image of the measure μ under the mapping $f \mapsto (f(s_1), f(s_2))$. $\square$

Proof of Theorem 5.2.3:
We recall that for a simple max-stable random field, the exponent measure μ is homogeneous of order -1, i.e. $\mu(cA) = c^{-1}\mu(A)$ for all $A \subset \mathbb{C}_0$ Borel set and $c > 0$. Also the assumption of standard unit Fréchet marginals implies

$$\bar{\mu}_t(y) = y^{-1}, \quad t \in T,\ y > 0.$$

These conditions imply (See Giné and al. [44] Propostition 3.2 or de Haan and Fereira [31] Theorem 9.4.1 and Corollary 9.4.2) that μ can be written as

$$\mu(A) = \int_0^\infty \int_{\mathcal{C}_0} 1_{\{rf \in A\}}\, r^{-2} dr \sigma(df)$$

where σ is a probability measure on $\mathbb{C}_0$ such that

$$\int_{\mathcal{C}_0} f(t)\, \sigma(df) = 1 \quad \text{for all } t \in T,$$

and

$$\int_{\mathcal{C}_0} \sup_{s \in S} f(s)\, \sigma(df) < \infty \quad \text{for all compact } S \subset T.$$

Using this, note that for all compact $S \subset T$ and $y > 0$,

$$\begin{aligned}
\mathbb{P}[\sup_{s \in S} f(s) \leq y] &= \exp[-\mu(\{f \in \mathbb{C}_0;\ \sup_{s \in S} f(s) > y\})] \\
&= \exp\Big[-\int_{\mathcal{C}_0} 1_{\{\sup_{s \in S} rf(s) > y\}}\, r^{-2} dr \sigma(df) \Big] \\
&= \exp\Big[-y^{-1} \int_{\mathcal{C}_0} \sup_{s \in S} f(s)\, \sigma(df) \Big].
\end{aligned}$$

It follows that the extremal coefficient $\theta(S)$ defined by (5.4) is equal to

$$\theta(S) = \int_{\mathcal{C}_0} \sup_{s \in S} f(s)\, \sigma(df). \tag{5.19}$$

We now consider $C(S)$ defined by equation (5.5). Since $S \subset T$ compact,

$$C(S) = \mathbb{E}\Big[(\inf_{s\in S} \eta(s))^{-1}\Big],$$

we need to provide a lower bound for $\inf_{s\in S} \eta(s)$. To this aim, we remark that

$$\inf_{s\in S} \eta(s) = \inf_{s\in S} \max_{\phi\in\Phi} \phi(s) \geq \max_{\phi\in\Phi} \inf_{s\in S} \phi(s).$$

The right hand side is a random variable with unit Fréchet distribution since

$$\begin{aligned}
\mathbb{P}[\max_{\phi\in\Phi} \inf_{s\in S} \phi(s) \leq y] &= \mathbb{P}[\forall \phi \in \Phi,\ \inf_{s\in S} \phi(s) \leq y] \\
&= \exp(-\mu(\{f \in \mathbb{C}_0;\ \inf_{s\in S} f(s) > y\}))
\end{aligned}$$

and

$$\mu(\{f \in \mathbb{C}_0;\ \inf_{s\in S} f(s) > y\}) = y^{-1} \int_{\mathcal{C}_0} \inf_{s\in S} f(s)\, \sigma(df).$$

Hence, if $\int_{\mathcal{C}_0} \inf_{s\in S} f(s)\, \sigma(df) > 0$, we obtain

$$C(S) = \mathbb{E}\Big[(\inf_{s\in S} \eta(s))^{-1}\Big] \leq \mathbb{E}\Big[(\max_{\phi\in\Phi} \inf_{s\in S} \phi(s))^{-1}\Big] = \Big(\int_{\mathcal{C}_0} \inf_{s\in S} f(s)\, \sigma(df)\Big)^{-1} < \infty.$$

For arbitrary compact $S \subset T$, we may however have $\int_{\mathcal{C}_0} \inf_{s\in S} f(s)\, \sigma(df) = 0$. But, if $S = B(s_0, \varepsilon)$ is a closed ball with center s_0 and radius ε, the monotone convergence Theorem implies

$$\int_{\mathcal{C}_0} \inf_{s\in B(s_0,\varepsilon)} f(s)\, \sigma(df) \to \int_{\mathcal{C}_0} f(s_0)\, \sigma(df) = 1 \quad \text{as } \varepsilon \to 0,$$

so that $\int_{\mathcal{C}_0} \inf_{s\in B(s_0,\varepsilon_0)} f(s)\, \sigma(df) > 0$ and $C(B(s_0, \varepsilon_0)) < \infty$ for ε_0 small enough. The result for general S follows by a compacity argument : there exists $s_1, \ldots, s_k$ and $\varepsilon_1, \ldots, \varepsilon_k$ such that $S \subset \cup_{i=1}^k B(s_i, \varepsilon_i)$. Hence,

$$\sup_{s\in S} \eta(s)^{-1} \leq \max_{1\leq i\leq k} \sup_{s\in B(s_i,\varepsilon_i)} \eta(s)^{-1} \leq \sum_{i=1}^k \sup_{s\in B(s_i,\varepsilon_i)} \eta(s)^{-1}$$

and

$$C(S) = \mathbb{E}\Big[\sup_{s\in S} \eta(s)^{-1}\Big] \leq \sum_{i=1}^k \mathbb{E}\Big[\sup_{s\in B(s_i,\varepsilon_i)} \eta(s)^{-1}\Big] = \sum_{i=1}^k C(B(s_i, \varepsilon_i)) < \infty.$$

This proves the fact that $C(S)$ is finite.

– The upper bound for $\beta(S_1, S_2)$ given by Theorem 5.2.1 can be expressed as

$$\begin{aligned}
\beta(S_1,S_2) &\leq 2\int_{\mathcal{C}_0} \mathbb{P}[f \not<_{S_1} \eta,\ f \not<_{S_2} \eta]\,\mu(df) \\
&= 2\int_{\mathcal{C}_0}\int_0^\infty \mathbb{P}[rf \not<_{S_1} \eta,\ rf \not<_{S_2} \eta]\,r^{-2}dr\sigma(df) \\
&= 2\int_{\mathcal{C}_0}\int_0^\infty \mathbb{E}\Big[1_{\{r\geq \inf_{s_1\in S_1}\eta(s_1)/f(s_1),\ r\geq \inf_{s_2\in S_2}\eta(s_2)/f(s_2)\}}\Big]\,r^{-2}dr\sigma(df) \\
&= 2\int_{\mathcal{C}_0} \mathbb{E}\Big[\max\big(\inf_{s_1\in S_1}\frac{\eta(s_1)}{f(s_1)},\ \inf_{s_2\in S_2}\frac{\eta(s_2)}{f(s_2)}\big)^{-1}\Big]\,\sigma(df). \qquad (5.20)
\end{aligned}$$

We then introduce the upper bound

$$\begin{aligned}
&\max\Big(\inf_{s_1\in S_1}\frac{\eta(s_1)}{f(s_1)},\ \inf_{s_2\in S_2}\frac{\eta(s_2)}{f(s_2)}\Big)^{-1} \\
\leq\ &\max(\sup_{s_1\in S_1}\eta(s_1)^{-1},\ \sup_{s_2\in S_2}\eta(s_2)^{-1})\ \min(\sup_{s_1\in S_1}f(s_1),\ \sup_{s_2\in S_2}f(s_2))
\end{aligned}$$

whence we deduce

$$\begin{aligned}
&\beta(S_1,S_2) \\
=\ & 2\mathbb{E}\Big[\max\big(\sup_{s_1\in S_1}\eta(s_1)^{-1},\ \sup_{s_2\in S_2}\eta(s_2)^{-1}\big)\Big]\int_{\mathcal{C}_0}\min(\sup_{s_1\in S_1}f(s_1),\ \sup_{s_2\in S_2}f(s_2))\,\sigma(df) \\
\leq\ & 2\mathbb{E}\Big[\sup_{s_1\in S_1}\eta(s_1)^{-1} + \sup_{s_2\in S_2}\eta(s_2)^{-1}\Big]\int_{\mathcal{C}_0}\min(\sup_{s_1\in S_1}f(s_1),\ \sup_{s_2\in S_2}f(s_2))\,\sigma(df) \\
=\ & 2\big[C(S_1)+C(S_2)\big]\ \big[\theta(S_1)+\theta(S_2)-\theta(S_1\cup S_2)\big].
\end{aligned}$$

In the last equality, we use Equation (5.5) defining $C(S)$ and Equation (5.19) defining $\theta(S)$ together with the following simple equality

$$\min(\sup_{s_1\in S_1}f(s_1),\ \sup_{s_2\in S_2}f(s_2)) + \max(\sup_{s_1\in S_1}f(s_1),\ \sup_{s_2\in S_2}f(s_2)) = \sup_{s_1\in S_1}f(s_1) + \sup_{s_2\in S_2}f(s_2).$$

– The second point is straightforward since

$$\begin{aligned}
\mathbb{P}[f \not<_{S_1} \eta,\ f \not<_{S_2} \eta] &= \mathbb{P}[\exists i\in I,\ f \not<_{S_{1,i}} \eta,\ \exists j\in J,\ f \not<_{S_{2,j}} \eta] \\
&\leq \sum_{i\in I}\sum_{j\in J}\mathbb{P}[f \not<_{S_{1,i}} \eta,\ f \not<_{S_{2,j}} \eta]
\end{aligned}$$

so that

$$\begin{aligned}
\beta(S_1,S_2) &\leq \sum_{i\in I}\sum_{j\in J}\int_{\mathcal{C}_0}\mathbb{P}[f \not<_{S_{1,i}} \eta,\ f \not<_{S_{2,j}} \eta]\,\mu(df) \\
&\leq 2\sum_{i\in I}\sum_{j\in J}\big[C(S_{1,i})+C(S_{2,j})\big]\ \big[\theta(S_{1,i})+\theta(S_{2,j})-\theta(S_{1,i}\cup S_{2,j})\big].
\end{aligned}$$

□

Proof of Corollary 5.2.4:
This is a straightforward consequence of the second point of Theorem 5.2.3 with $S_1 = \cup_{s_1 \in S_1}\{s_1\}$ and $S_2 = \cup_{s_2 \in S_2}\{s_2\}$. It holds indeed

$$\beta(S_1, S_2) \leq 2 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \left[C(\{s_1\}) + C(\{s_2\})\right] \left[\theta(\{s_1\}) + \theta(\{s_2\}) - \theta(\{s_1\} \cup \{s_2\})\right]$$

with

$$\theta(\{s_1\}) = \theta(\{s_2\}) = 1 \quad , \quad \theta(\{s_1\} \cup \{s_2\}) = \theta(s_1, s_2)$$

and

$$C(\{s_1\}) = C(\{s_2\}) = 1.$$

The last equality follows from the fact that, for all $s \in S$, $\eta(s)$ has a standard unit Fréchet distribution and hence $\eta(s)^{-1}$ has an exponential distribution with mean 1. Hence we obtain

$$\beta(S_1, S_2) \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} (2 - \theta(s_1, s_2))$$

□

5.3.4 Proof of Theorems 5.2.6 and Proposition 5.2.7

Proof of Theorem 5.2.6:
According to Bolthausen's CLT for stationary mixing random fields (see Appendix 5.A.3), it is enough to prove that the mixing coefficients $\alpha_{k,l}(m)$ defined by Equation (5.21) with $X(t) = g(\eta(t_1 + t), \ldots, \eta(t_p + t))$ satisfy Equations (5.22), (5.23) and (5.24).

For $S \subset \mathbb{Z}^d$, we define $\widetilde{S} = \cup_{i=1}^{p}\{s + t_i,\ s \in S\}$. The inclusion of σ-fields

$$\sigma(\{X(s),\ s \in S\}) \subset \sigma(\{\eta(s),\ s \in \widetilde{S}\})$$

entails a comparison of the related α-mixing coefficients : for disjoint $S_1, S_2 \subset \mathbb{Z}$,

$$\alpha^X(S_1, S_2) \leq \alpha^\eta(\widetilde{S}_1, \widetilde{S}_2),$$

where the superscript X or η denotes that we are computing the α-mixing coefficient of the random field X or η respectively. Furthermore,

$$|\widetilde{S}_i| \leq p|S_i|,\ i = 1, 2 \quad \text{and} \quad d(\widetilde{S}_1, \widetilde{S}_2) \geq d(S_1, S_2) - \Delta,$$

with $\Delta = \max_{1 \leq i < j \leq p} d(t_i, t_j)$ the diameter of $\{t_1, \ldots, t_p\}$. Hence, with obvious notations,

$$\alpha_{k,l}^X(m) \leq \alpha_{pk,pl}^\eta(m - \Delta), \quad k, l \in \mathbb{N} \cup \{\infty\},\ m \geq \Delta + 1.$$

Similar to the proof of Corollary 5.2.4, we get

$$\alpha^X_{k,l}(m) \leq \alpha^{\eta}_{pk,pl}(m-\Delta) \leq p^2 kl \sup_{|t|\geq m-\Delta} \gamma(t), \quad k,l \in \mathbb{N},\ m \geq \Delta+1,$$

and

$$\alpha^X_{1,\infty}(m) \leq \alpha^{\eta}_{p,\infty}(m-\Delta) \leq p \sum_{|t|\geq m-\Delta} \gamma(t), \quad m \geq \Delta+1.$$

In view of this, Assumption (5.6) entails Equations (5.22), (5.23) and (5.24), so that the random field X satisfies Bolthausen's CLT. $\square$

Proof of Proposition 5.2.7:
Let $h \in \mathbb{Z}^d$. We apply Theorem 5.2.6 to the stationary random field

$$X(t) = 1_{\{\eta(t)\leq y,\ \eta(t+h)\leq y\}}, \quad t \in \mathbb{Z}^d.$$

Clearly $\mathbb{E}[X(t)] = \exp[-\theta(h)/y]$ and $\mathbb{E}[|X|^{2+\delta}] < \infty$ for all $\delta > 0$. Assumption (5.9) together with $\gamma(t) \leq 4(2-\theta(t))$ ensure that Equation (5.6) is satisfied for δ large enough. Hence the estimator $\hat{p}_n(h,y)$ is asymptotically normal :

$$|\Lambda_n|^{1/2}\Big(\hat{p}_n(h,y) - p(h,y)\Big) \Longrightarrow \mathcal{N}(0,\beta_1^2)$$

with limit variance

$$\begin{aligned} \beta_1^2(y) &= \sum_{t\in\mathbb{Z}^d} \mathrm{Cov}[X(0),X(t)] \\ &= \sum_{t\in\mathbb{Z}^d} \Big(\exp(\theta(\{0,h,t,t+h\})/y) - \exp(2\theta(h))/y)\Big) > 0. \end{aligned}$$

The δ-method entails the asymptotic normality of the estimator $\hat{\theta}^y_n(h) = -y\log\hat{p}_n(h,y)$:

$$|\Lambda_n|^{1/2}\Big(\hat{\theta}^y_n(h) - \theta(h)\Big) \Longrightarrow \mathcal{N}(0,\sigma_1^2)$$

with limit variance

$$\sigma_1^2 = y^2\exp(2\theta(h)/y)\beta_1^2 = y^2 \sum_{t\in\mathbb{Z}^d} \Big(\exp[(2\theta(h) - \theta(\{0,h,t,t+h\}))/y] - 1\Big).$$

The proof of the asymptotic normality of $\hat{\theta}^{(2)}_n$ and $\hat{\theta}^{(3)}_n$ is very similar and we give only the main lines. Using Theorem 5.2.6, we prove that

$$\hat{q}_n = |\Lambda_n|^{-1} \sum_{t\in|\Lambda_n|} \min(\eta(t)^{-1}, \eta(t+h)^{-1})$$

is an asymptotic normal estimator of $\theta(h)^{-1}$:

$$|\Lambda_n|^{1/2}(\hat{q}_n - \hat{\theta}(h)^{-1}) \Longrightarrow \mathcal{N}(0, \beta_2^2)$$

with limit variance

$$\beta_2^2 = \sum_{t \in \mathbb{Z}^d} \mathrm{Cov}\big[\min(\eta(0)^{-1}, \eta(h)^{-1}), \min(\eta(t)^{-1}, \eta(t+h)^{-1})\big].$$

The δ-method entails the asymptotic normality of $\theta_n^{(2)}(h) = 1/\hat{q}_n(h)$ with limit variance

$$\sigma_2^2 = \theta(h)^4 \beta_1^2.$$

Similarly,

$$\hat{\nu}_{F,n}(h) = |\Lambda_n|^{-1} \sum_{t \in |\Lambda_n|} |F(\eta(t)) - F(\eta(t+h))|$$

is an asymptotic normal estimator of $\nu_F(h) = \mathbb{E}[|F(\eta(0)) - F(\eta(h))|]$:

$$|\Lambda_n|^{1/2}(\hat{\nu}_{F,n}(h) - \nu_F(h)) \Longrightarrow \mathcal{N}(0, \beta_3^2)$$

with limit variance

$$\beta_3^2 = \sum_{t \in \mathbb{Z}^d} \mathrm{Cov}\big[|F(\eta(0)) - F(\eta(h))|, |F(\eta(t)) - F(\eta(t+h))|\big].$$

The δ-method entails the asymptotic normality of

$$\theta_n^{(3)}(h) = \frac{1 + 2\hat{\nu}_{F,n}(h)}{1 - 2\hat{\nu}_{F,n}(h)}$$

with limit variance

$$\sigma_3^2 = (\theta(h) + 1)^4 \beta_3^2.$$

□

APPENDIX

5.A Auxiliary results

5.A.1 Structure of max-i.d. random processes

The structure of sample continuous random processes on a compact metric space was elucidated by Hahn, Giné and Vatan [44]. Further results by Vatan [76] cover the discrete case $T = \mathbb{Z}^d$. We give here similar results when T is a locally compact metric space, typically $T = \mathbb{R}^d$. Such extensions have been considered for max-stable models on $\mathbb{R}$ (see [31] Chapter 9.6) but we have found no reference in the max-i.d. case.

Let η be a continuous max-i.d. random process on $\mathbb{C}(T, \mathbb{R})$. Define its vertex function $h : T \to [-\infty, +\infty)$ by

$$h(t) = \text{essinf}\, \eta(t) = \sup\{x \in \mathbb{R};\ \mathbb{P}(\eta(t) \geq x) = 1\}.$$

We will always assume that h is continuous. We can then suppose without loss of generality that $h \equiv 0$. Indeed, if h is continuous and finite, we may consider $\eta - h$ which is a continuous max-i.d. random field with zero vertex function ; and if h is not finite everywhere, we may consider $\exp(\eta) - \exp(h)$ which is max-i.d with zero vertex function.

We denote by $\mathbb{C}(T) = \mathbb{C}(T, [0, +\infty))$ the space of nonnegative continuous functions on T endowed with the topology of uniform convergence on compact sets and set $\mathbb{C}_0(T) = \mathbb{C}(T)\backslash\{0\}$.

Theorem 5.A.1. – *Let $\eta = (\eta(t))_{t\in T}$ be a continuous max-i.d. process on T with vertex function $h \equiv 0$. There exists a unique locally finite Borel measure on $\mathbb{C}_0$ satisfying condition* (5.1)*, called the exponent measure of η, such that*

$$\log \mathbb{P}\Big[\cap_{i=1}^k \{\eta(t_i) \leq y_i\}\Big] = -\mu\Big[\cup_{i=1}^k \{f \in \mathbb{C}_0; f(t_i) > y_i\}\Big]$$

for all $k \geq 1$, $t_1, \ldots, t_k \in T$ and $y_1, \ldots, y_k > 0$.

– *Conversely, for any locally finite Borel measure on $\mathbb{C}_0$ satisfying condition* (5.1)*, there exists a continuous max-i.d. process η on T with vertex function $h \equiv 0$ and exponent measure μ. It can be constructed as follows : let Φ be a Poisson point process on $\mathbb{C}_0$ with intensity μ and define*

$$\eta(t) = \max\{\phi(t),\ \phi \in \Phi\}, \quad t \in T.$$

Proof of Theorem 5.A.1:
Let $(T_n)_{n\geq 1}$ be an increasing sequence of compact sets such that $T = \cup_{n\geq 1} T_n$. We suppose furthermore that T_n is included in the interior set of T_{n+1}. The space $\mathbb{C}(T) = \mathbb{C}(T, \mathbb{R}^+)$ of nonnegative continuous functions on T endowed with the topology of uniform convergence on compact sets can be seen as the projective limit of the sequence of spaces $\mathbb{C}(T_n, \mathbb{R}^+)$ endowed with the topology of uniform convergence. For $m \geq n \geq 1$, we define the natural projections

$$\pi_n : \mathbb{C}(T) \to \mathbb{C}(T_n) \quad \text{and} \quad \pi_{n,m} : \mathbb{C}(T_m) \to \mathbb{C}(T_n).$$

For each $n \geq 1$, the restriction $\pi_n(\eta) = \eta_{T_n}$ is a continuous max-i.d. process on the compact space T_n and according to [44], there exists a locally finite exponent measure μ_n on $\mathbb{C}_0(T_n) = \mathbb{C}(T_n) \setminus \{0\}$ satisfying equation

$$\log \mathbb{P}\Big[\cap_{i=1}^k \{\eta(t_i) \leq y_i\}\Big] = -\mu_n\Big[\cup_{i=1}^k \{f \in \mathbb{C}_0; f(t_i) > y_i\}\Big]$$

for all $k \geq 1$, $t_1, \ldots, t_k \in T_n$ and $y_1, \ldots, y_k > 0$. Furthermore, for all $\varepsilon > 0$

$$\mu_n[\mathcal{S}_{n,\varepsilon}] < \infty \quad \text{where} \quad \mathcal{S}_{n,\varepsilon} = \Big\{f \in \mathbb{C}(T_n);\ \sup_{T_n} f > \varepsilon\Big\}.$$

Let $n_0 \geq 1$ and $\varepsilon > 0$ be fixed. For $n \geq n_0$, define the finite Borel measure by

$$\tilde{\mu}_n^{n_0,\varepsilon}[A] = \mu_n[A \cap \pi_{n_0,n}^{-1}\mathcal{S}_{n_0,\varepsilon}], \quad A \subset \mathbb{C}(T_n) \text{ Borel set.}$$

Clearly, the following compatibility conditions holds true : for $m \geq n \geq n_0$,

$$\tilde{\mu}_n^{n_0,\varepsilon} = \tilde{\mu}_m^{n_0,\varepsilon}\pi_{n,m}^{-1}.$$

Note that since $\mathbb{C}_0(T)$ is a Polish space, every locally finite measure is inner regular and hence a Radon measure. Theorem 5.1.1 in [9] state the existence of projective limits of Radon measures, it implies the existence of a finite Radon measure $\tilde{\mu}^{n_0,\varepsilon}$ on $\mathbb{C}(T)$ such that

$$\tilde{\mu}_n^{n_0,\varepsilon} = \tilde{\mu}^{n_0,\varepsilon}\pi_n^{-1}, \quad n \geq n_0.$$

It is then easily checked that the measure μ on $\mathbb{C}_0(T)$ defined by

$$\mu[A] = \sup\{\tilde{\mu}^{n_0,\varepsilon}[A];\ n_0 \geq 1,\ \varepsilon > 0\}, \quad A \subset \mathbb{C}_0(T) \text{ Borel set}$$

is locally finite and enjoys the required properties. □

5.A.2 Slyvniak's formula

Palm Theory deals with conditional distribution for point processes. We recall here one of the most famous formula of Palm theory, known as Slyvniak's Theorem. This will be the main tool in our computations. For a general reference on Poisson point processes, Palm theory and their applications, the reader is invited to refer to the monograph [74] by Stoyan, Kendall and Mecke.

The following formula is obtained thanks to Campbell's Theorem and Slyvniak's Theorem together, and is sometimes reffered to as Campbell-Slyvniak formula. For our purpose, we state it for $\mathbb{C}_0$ valued point processes. Let $M_p(\mathbb{C}_0)$ be the set of locally finite point measures N on $\mathbb{C}_0$ endowed with the σ-algebra generated by the family of mappings $\{N \mapsto N(A),\ A \subset \mathbb{C}_0 \text{ Borel set}\}$.

Theorem 5.A.2 (Campbell-Slyvniak Formula)**.**
Let Φ be a Poisson point process on $\mathbb{C}_0$ with intensity measure μ. For all measurable function $F : \mathbb{C}_0^k \times M_p(\mathbb{C}_0) \to [0, +\infty)$,

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathcal{C}_0^k} F\Big(\phi_1, \ldots, \phi_k, \Phi - \sum_{i=1}^{k} \delta_{\phi_i}\Big)\, \Phi(d\phi_1)\,(\Phi - \delta_{\phi_1})(d\phi_2) \cdots (\Phi - \sum_{j=1}^{k-1} \delta_{\phi_j})(d\phi_k)\Big] \\
= &\int_{\mathcal{C}_0^k} \mathbb{E}[F(f_1, \ldots, f_k, \Phi)]\, \mu^{\otimes k}(df_1, \ldots, df_k).
\end{aligned}$$

5.A.3 A central limit Theorem for weakly dependent process

Since the pioneer work of Ibragimov [48], many versions of the central limit Theorem for weakly dependent processes have been developed under various strong mixing conditions. We present here a central limit Theorem for stationary mixing random fields due to Bolthausen [10]. Let $(X_k)_{k\in\mathbb{Z}^d}$ be a real valued stationary random field and recall the definition of the α-mixing coefficient (5.21). If $\Lambda \subset \mathbb{Z}^d$, we denote by $|\Lambda|$ the number of elements in Λ and by $\partial\Lambda$ the set of elements $k \in \Lambda$ such that there is $l \notin \Lambda$ with $d(k,l) = 1$. Let Λ_n be a fixed increasing sequence of finite subsets of $\mathbb{Z}^d$, which increases to $\mathbb{Z}^d$ and such that $\lim_{n\to\infty} |\partial\Lambda_n|/|\Lambda_n| = 0$. Let $\Sigma_n = \sum_{h\in\Lambda_n}(X_h - \mathbb{E}[X_h])$.

For subsets $S_1, S_2 \subset \mathbb{Z}^d$, we define

$$d(S_1,S_2) = \min\{|s_2 - s_1|;\ s_1 \in S_1,\ s_2 \in S_2\}.$$

Bolthausen's central limit Theorem is based on the mixing coefficients

$$\alpha_{kl}(m) = \sup\Big\{\alpha(S_1,S_2);\ |S_1| = k, |S_2| = l,\ d(S_1,S_2) \geq m\Big\} \tag{5.21}$$

defined for $m \geq 1$ and $k, l \in \mathbb{N}\cup\{\infty\}$.

Theorem 5.A.3. *Suppose that the following three conditions are satisfied :*

$$\alpha_{1\infty}(m) = o(m^{-d}); \tag{5.22}$$

$$\textstyle\sum_{m=1}^{\infty} m^{d-1}\alpha_{kl}(m) < \infty \quad \textit{for all}\ \ k \geq 1,\ l \geq 1 \textit{ such that } k+l \leq 4; \tag{5.23}$$

$$\mathbb{E}\big[|X_h|^{2+\delta}\big] < \infty \quad \textit{and} \quad \textstyle\sum_{m=1}^{\infty} m^{d-1}|\alpha_{11}(m)|^{\delta/(2+\delta)} < \infty \quad \textit{for some } \delta > 0. \tag{5.24}$$

Then the series $\sigma^2 = \sum_{h\in\mathbb{Z}^d}\mathrm{Cov}[X_0, X_h]$ converges absolutely and if furthermore $\sigma^2 > 0$,

$$\frac{\Sigma_n}{\sigma|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0,1), \quad \textit{as } n \to \infty.$$

Chapitre 6

Stationary max-stable processes with the Markov property

Abstract

We prove that the class of discrete time stationary max-stable process satisfying the Markov property is equal, up to time reversal, to the class of stationary max-autoregressive processes of order 1. A similar statement is also proved for continuous time processes.

6.1 Introduction

Given a class of stochastic processes, a natural and important question is to determine conditions ensuring the Markov property. For example, a zero mean Gaussian process X on $T = \mathbb{Z}$ or $\mathbb{R}$ satisfies the Markov property if and only if

$$\mathbb{E}[X(t_2) \mid X(t), t \leq t_1] = \mathbb{E}[X(t_2) \mid X(t_1)] \quad \text{for all } t_1, t_2 \in T, t_1 \leq t_2.$$

It is also well known that if X is a *stationary* zero mean Gaussian processes on T satisfying the Markov property, then X must be the Ornstein-Uhlenbeck process with covariance function

$$\mathbb{E}[X(t_1)X(t_2)] = \mathbb{E}[X(0)^2]e^{-\lambda|t_2-t_1|}$$

for some $\lambda \in [0, +\infty]$. The case $\lambda = 0$ corresponds to a constant process, the case $\lambda = +\infty$ to a Gaussian white noise.

Within the class of symmetric α-stable (SαS) processes, the situation is much more complicated (see Adler *et al.* [1]). No complete characterization of SαS Markov processes is known but only necessary or sufficient conditions. One can construct at least two classes of stationary Markov SαS processes, the right and the left SαS Ornstein-Uhlenbeck processes.

The purpose of this paper is to study the Markov property within the class of max-stable random processes. Without loss of generality, we shall consider only *simple* max-stable processes defined as follows.

Definition 6.1.1. *A random process $\eta = (\eta(t))_{t\in T}$ is said to be simple max-stable if it has* 1-*Fréchet marginals*

$$\mathbb{P}[\eta(t) \leq y] = \exp(-1/y), \quad \textit{for all } t \in T,\ y > 0,$$

and satisfies the following max-stability property :

$$n^{-1} \bigvee_{i=1}^{n} \eta_i \stackrel{d}{=} \eta, \quad \textit{for all } n \geq 1,$$

where $(\eta_i)_{i\geq 1}$ are independent copies of η and $\bigvee$ denotes pointwise maximum, $\stackrel{d}{=}$ denotes the equality of distributions.

Our main result is a complete characterization of the class of stationary simple max-stable Markov processes on $T = \mathbb{Z}$ or $\mathbb{R}$. Our analysis relies on a recent paper [35] where explicit formulas for the conditional distributions of max-stable processes are proved. This helps clarifying the notion of (Markovian) dependence for max-stable processes.

Related works by Tavares [75], Alpuim [2], Alpuim *et al.* [3] characterized stationary max-AR(1) processes, and Alpuim *et al.* [4] study max-autoregressive processes and the Markov property in extreme value theory. Extremes of Markov chains have been considered by Perfekt [60] and Smith [71], while Smith *et al.* [72] consider Markov chain models for threshold exceedances (see the monograph by Beirlant *et al.* section 10.4 for further discussion on extremes and Markov chains).

Well known examples of discrete time simple max-stable processes satisfying the Markov property are maximum-autoregressive processes of order 1. The max-AR(1) process with parameter $a \in [0,1]$ is defined as follows : consider $(F_n)_{n\in\mathbb{Z}}$, a sequence of i.i.d. random variables with standard 1-Fréchet distribution, and set

$$X_a(t) = \begin{cases} \bigvee_{n\leq t} (1-a)a^{t-n} F_n & \text{if } a \in [0,1) \\ X_1(t) \equiv F_0 & \text{if } a = 1 \end{cases}, \quad t \in \mathbb{Z}. \tag{6.1}$$

The max-AR(1) process X_a is a stationary simple max-stable process satisfying

$$X_a(t+1) = \max(aX_a(t), (1-a)F_{t+1}), \quad t \in \mathbb{Z}.$$

This relation explains the term max-autoregressive and implies that X_a satisfies the Markov property. The associated Markov kernel $K_a(x,\cdot)$ is defined by

$$K_a(x,dy) = \mathbb{P}[X_a(t+1) \in dy \mid X_a(t) = x]$$

and is easily computed : denoting δ_z the Dirac measure at point z, it holds

$$K_a(x,dy) = e^{-(1-a)/(ax)} \delta_{ax}(dy) + (1-a) y^{-2} e^{-(1-a)/y} 1_{\{y>ax\}} dy.$$

Note that the parameter $a \in [0,1]$ tunes the strength of the dependence, ranging from independence when $a = 0$ to complete dependence when $a = 1$. It can be retrieved from the support of the law of $X_a(t+1)/X_a(t)$ since

$$\operatorname{supp}(X_a(t+1)/X_a(t)) = [a, +\infty) \quad \text{if } a \in [0,1),$$

and the support is reduced to $\{1\}$ in the case $a = 1$.

It is well known that if $X = (X(t))_{t\in\mathbb{Z}}$ is a stationary Markov chain, then the time reversed process $\check{X} = (X(-t))_{t\in\mathbb{Z}}$ is also a stationary Markov chain. Hence, time reversed max-autoregressive processes are further examples of stationary max-stable Markov processes. More precisely, if X_a is the max-AR(1) process (6.1), the associated time reversed process $\check{X}_a$ is given by

$$\check{X}_a(t) = (1-a) \bigvee_{n\geq t} a^{n-t} \check{F}_n, \quad t \in \mathbb{Z}, \tag{6.2}$$

where $(\check{F}_n)_{n\in\mathbb{Z}} = (F_{-n})_{n\in\mathbb{Z}}$ are i.i.d. random variables with standard 1-Fréchet distribution. Clearly, $\check{X}_a$ satisfies the backward max-autoregressive relation

$$\check{X}_a(t-1) = \max(a\check{X}_a(t), (1-a)\check{F}_{t-1}), \quad t \in \mathbb{Z}.$$

The Markov kernel associated to $\check{X}_a$ is given by

$$\begin{aligned} \check{K}_a(y, dx) &= \mathbb{P}[\check{X}_a(t+1) \in dx \mid \check{X}_a(t) = y] \\ &= a\delta_{y/a}(dx) + (1-a)x^{-2}e^{-1/x+a/y}1_{\{x<y/a\}}dx. \end{aligned}$$

Note that the Markov kernels K_a and $\check{K}_a$ are related by the equilibrium relation

$$\pi(dx)K_a(x, dy) = \pi(dy)\check{K}_a(y, dx)$$

where $\pi(dx) = x^{-2}e^{-1/x}1_{\{x>0\}}dx$ is the stationary distribution. It is easily seen that for $a = 0$ or $a = 1$, X_a and $\check{X}_a$ have the same distribution. This means that the max-AR(1) process X_a is reversible if $a = 0$ or $a = 1$. This is no longer the case when $a \in (0,1)$ since

$$\operatorname{supp}(\check{X}_a(t+1)/\check{X}_a(t)) = [0, 1/a] \quad \text{if } a \in (0,1).$$

The purpose of the present paper is the characterization of all stationary simple max-stable processes satisfying the Markov property.

Theorem 6.1.2. *Any stationary simple max-stable process $\eta = (\eta(t))_{t\in\mathbb{Z}}$ satisfying the Markov property is equal in distribution to a max-AR(1) process* (6.1) *or to a time-reversed max-AR(1) process* (6.2).

The structure of the paper is the following. In section 2, we gather some preliminaries on max-stable processes and their representations that will be useful in our approach. Section 3 is devoted to the proof of Theorem 6.1.2. An extension to continuous time processes is considered in section 4.

6.2 Preliminaries on max-stable processes

6.2.1 Representations of max-stable processes

Our approach relies on the following representation of simple max-stable process due to de Haan [30], see also Penrose [59] and Schlather [67]. The symbol $\stackrel{d}{=}$ stands for equality in distribution.

Theorem 6.2.1. *Let $\eta = (\eta(t))_{t\in\mathbb{Z}}$ be a simple max-stable process on $\mathbb{Z}$. Then, there exists a nonnegative random process Y such that*

$$\mathbb{E}[Y(t)] = 1 \quad \textit{for all } t \in \mathbb{Z}, \tag{6.3}$$

and

$$\Big(\eta(t)\Big)_{t\in\mathbb{Z}} \stackrel{d}{=} \Big(\bigvee_{i\geq 1} U_i Y_i(t)\Big)_{t\in\mathbb{Z}}, \tag{6.4}$$

where $(Y_i)_{i\geq 1}$ are i.i.d. copies of Y and $\{U_i, i \geq 1\}$ is a Poisson point process on $(0,+\infty)$ with intensity $u^{-2}du$ and independent of $(Y_i)_{i\geq 1}$.

The random process Y is called a spectral process associated to η. Conversely, we call η the max-stable process associated to Y.

Consider the function space $\mathcal{F} = [0,+\infty)^{\mathbb{Z}}$ endowed with the product sigma-algebra and $\mathcal{F}_0 = \mathcal{F} \setminus \{0\}$. The exponent measure of η is defined by

$$\mu(A) = \int_0^\infty \mathbb{P}[uY \in A] u^{-2} dy, \quad A \in \mathcal{F}_0 \text{ measurable}. \tag{6.5}$$

It does not depend on the choice of the spectral process Y but only on the distribution of η. It satisfies the homogeneity property

$$\mu(uA) = u^{-1}\mu(A), \quad u > 0,\ A \in \mathcal{F}_0 \text{ measurable},$$

and is related to η by the relations

$$\begin{aligned} &\mathbb{P}[\eta(t_1) \leq z_1, \ldots, \eta(t_k) \leq z_k] \\ = \ &\exp\Big(-\mu\{f \in \mathcal{F};\ f(t_i) > z_i \text{ for some } 1 \leq i \leq k\}\Big) \end{aligned}$$

for all $k \geq 1$, $t_1, \ldots, t_k \in \mathbb{Z}$, $z_1, \ldots, z_k \geq 0$.

Note that there is no uniqueness for the representation (6.4). We introduce therefore the following notion of equivalent spectral processes.

Definition 6.2.2. *Let Y and Y' be nonnegative stochastic processes satisfying*

$$\mathbb{E}[Y(t)] = \mathbb{E}[Y'(t)] = 1, \quad t \in \mathbb{Z}. \tag{6.6}$$

We say that Y and Y' are equivalent if and only if the associated max-stable processes have the same distribution.

The following property will be useful. A subset $C \subset \mathcal{F}$ is called a cone if and only if $f \in C$ implies $uf \in C$ for all $u \geq 0$.

Proposition 6.2.3. *Let Y and Y' be equivalent processes as in Definition 6.2.2. Let $C \subset \mathcal{F}$ be a measurable cone such that $\mathbb{P}[Y \in C] = 1$. Then, $\mathbb{P}[Y' \in C] = 1$.*

Proof :
Let μ (resp. μ') be the exponent measure of the max-stable process associated to Y (resp. Y') by Equation (6.5). Clearly, Y and Y' are equivalent if and only the exponent measures μ and μ' are equal. On the other hand, Equation (6.5) implies clearly that $\mathbb{P}[Y \in C] = 1$ if and only if μ is supported by C, i.e. $\mu[\mathcal{F} \setminus C] = 0$. Similarly $\mathbb{P}[Y' \in C] = 1$ if and only if $\mu'[\mathcal{F} \setminus C] = 0$.

Using this, we deduce easily that if Y and Y' are equivalent processes with $\mathbb{P}[Y \in C] = 1$, then $\mu = \mu'$ is supported by C and $\mathbb{P}[Y' \in C] = 1$. $\square$

6.2.2 Brown-Resnick stationary processes

In the following we focus on stationary max-stable processes. A random process $X = (X(t))_{t\in\mathbb{Z}}$ is called stationary if X and $X(\cdot + s)$ have the same distribution for all $s \in \mathbb{Z}$. We use the following terminology, due to Kabluchko *et al.* [51].

Definition 6.2.4. *A nonnegative random process Y satisfying* (6.3) *is called Brown-Resnick stationary if the associated max-stable process η defined by* (6.4) *is stationary.*

It follows from the definition that Y is Brown-Resnick stationary if and only if Y and $Y(\cdot+s)$ are equivalent (in the sense of Definition 6.2.2) for all $s \in \mathbb{Z}$. Proposition 6.2.3 implies then the following result.

Proposition 6.2.5. *Let Y be a Brown-Resnick stationary process and let $C \subset \mathcal{F}$ be a measurable cone such that $\mathbb{P}[Y \in C] = 1$. Then, for all $s \in \mathbb{Z}$, $\mathbb{P}[Y(\cdot + s) \in C] = 1$. Furthermore, noting $\theta_s : \mathcal{F} \to \mathcal{F}$ the shift operator defined by $\theta_s(f) = f(\cdot + s)$, it holds*

$$\mathbb{P}\Big[Y \in \bigcap_{s\in\mathbb{Z}} \theta_s(C)\Big] = 1.$$

Proof :
As we noticed, if Y is Brown-Resnick stationary, then Y and $Y(\cdot + s)$ are equivalent for all $s \in \mathbb{Z}$. The result follows then directly from Proposition 6.2.3 by setting $Y' = Y(\cdot + s)$. For the last statement, if $\mathbb{P}[Y \in C] = 1$, then $\mathbb{P}[Y \in \theta_s(C)] = 1$ for all $s \in \mathbb{Z}$, whence we deduce $\mathbb{P}\big[Y \in \cap_{s\in\mathbb{Z}}\theta_s(C)\big] = 1$. $\square$

The following lemma will also be useful in order to prove equivalence of processes. For $f_0 \in \mathcal{F}$, we note $C_{inv}(f_0) = \{uf_0(\cdot + s);\ u \geq 0, s \in \mathbb{Z}\}$ the smallest shift-invariant cone containing f_0.

Lemma 6.2.6. *Let Y and Y' be Brown-Resnick stationary processes satisfying* (6.6) *and such that*

$$\mathbb{P}[Y \in C_{inv}(f_0)] = \mathbb{P}[Y' \in C_{inv}(f_0)] = 1 \quad \textit{for some } f_0 \in \mathcal{F}.$$

Then Y and Y' are equivalent.

Proof :
We note μ and μ' the exponent measure of the max-stable processes associated to Y and Y' respectively. Clearly the measure μ (and also μ') satisfies the following four properties :

i) μ is -1-homogeneous ;
ii) $\mu(\{f \in \mathcal{F};\ f(0) \geq 1\}) = 1$;
iii) μ is shift-invariant ;
iv) μ is supported by $C_{inv}(f_0)$.

Recall indeed that properties i) and ii) are satisfied for all exponent measures, that iii) holds if and only if the spectral process Y is Brown-Resnick stationary and that iv) is equivalent to $\mathbb{P}[Y \in C_{inv}(f_0)] = 1$.

We will prove below that there exists at most one measure μ on $\mathcal{F}_0$ satisfying the four properties i)-iv). Since μ' satisfies the same properties, we deduce that $\mu = \mu'$, whence Y and Y' are equivalent.

For $g \in \mathcal{F}$, we note $C(g) = \{ug;\ u \geq 0\}$ the smallest cone containing g. Clearly, $C_{inv}(f_0) = \cup_{s\in\mathbb{Z}} C(f_s)$ with $f_s = \theta_s f_0$. The different cones in the union may have non trivial intersections and two cases occur.

– First case : there is $s_0 \geq 1$ such that $C(f_{s_0}) \cap C(f_0) \neq \{0\}$.
Without loss of generality, we can suppose that s_0 is minimal with this property. Then,

$$C_{inv}(f) = \cup_{0\leq s\leq s_0-1} C(f_s)$$

and

$$C(f_s) \cap C(f_{s'}) = \{0\}, \quad 0 \leq s \neq s' \leq s_0 - 1.$$

– Second case : for all $s \geq 1$, $C(f_s) \cap C(f) = \{0\}$.
Then

$$C_{inv}(f) = \cup_{s\in\mathbb{Z}} C(f_s) \quad \text{and} \quad C(f_s) \cap C(f_{s'}) = \{0\}, \quad s \neq s'.$$

We give the proof in the first case only, the second case follows from straightforward modifications. The support property iv) implies that $\mu = \sum_{s=0}^{s_0-1} \mu_s$ where μ_s is the restriction of μ to $C(f_s)$. The homogeneity property i) entails that the restriction μ_s is completely determined by the real parameter $\alpha_s = \mu(\{uf_s; u \geq 1\})$. It holds indeed

$$\mu_s(\{uf_s; u \geq v\}) = v^{-1}\mu_s(\{uf_s; u \geq 1\}) = \alpha_s v^{-1}, \quad v > 0.$$

Furthermore, the shift invariance property iii) implies that $\alpha_s \equiv \alpha$ does not depend on s. Finally, the real parameter α is determined by the normalization property ii) : we have indeed

$$\mu(\{f \in \mathcal{F}; f(0) \geq 1\}) = \sum_{s=0}^{s_0-1} \mu_s(\{f \in \mathcal{F}; f(0) \geq 1\}) = \alpha \sum_{s=0}^{s_0-1} f_0(s)$$

whence property ii) yields $\alpha = (\sum_{s=0}^{s_0-1} f_0(s))^{-1}$. This proves that μ is completely determined by properties i)-iv) and completes the proof of the lemma. □

6.2.3 Conditional distributions

Our study of the Markov property for max-stable process relies on explicit formulas for the regular conditional distributions of max-stable process established in Dombry and Eyi-Minko [35]. The following expression for the conditional distribution function will be useful (see Proposition 4.1 in [35]).

Proposition 6.2.7. *Let η be a simple max-stable process with representation* (6.4). *For every $t, t_1, \ldots, t_k \in \mathbb{Z}$ and $z, z_1, \ldots, z_k > 0$*

$$\begin{aligned} & \mathbb{P}[\eta(t_1) \le z_1, \ldots, \eta(t_k) \le z_k \mid \eta(t) = z] \\ = \quad & \mathbb{E}\big[1_{\{\vee_{i=1}^k \frac{Y(t_i)}{z_i} \le \frac{Y(t)}{z}\}} Y(t)\big] \exp\Big(- \mathbb{E}\Big[\Big(\bigvee_{i=1}^k \frac{Y(t_i)}{z_i} - \frac{Y(t)}{z}\Big)^+\Big]\Big), \end{aligned}$$

with $(x)^+ = \max(x, 0)$.

We can deduce the following well known result, that we prove here for the sake of completeness.

Proposition 6.2.8. *Let η be a simple max-stable process with representation* (6.4) *and consider $t_1, t_2 \in \mathbb{Z}$. Then $\eta(t_1)$ and $\eta(t_2)$ are independent if and only if*

$$\mathbb{P}[Y(t_1) = 0 \textit{ or } Y(t_2) = 0] = 1.$$

Proof :
The random variables $\eta(t_1)$ and $\eta(t_2)$ are independent if and only if

$$\mathbb{P}[\eta(t_1) \le z_1 \mid \eta(t_2) = z_2] = \mathbb{P}[\eta(t_1) \le z_1], \quad z_1, z_2 > 0.$$

Using Proposition 6.2.7 and the fact that $\eta(t_1)$ has a standard 1-Fréchet distribution, this is equivalent to

$$\mathbb{E}\big[1_{\{\frac{Y(t_1)}{z_1} \le \frac{Y(t_2)}{z_2}\}} Y(t_2)\big] \exp\Big(- \mathbb{E}\Big[\Big(\frac{Y(t_1)}{z_1} - \frac{Y(t_2)}{z_2}\Big)^+\Big]\Big) = \exp\Big(- \frac{1}{z_1}\Big)$$

for all $z_1, z_2 > 0$. Setting $z = z_1$ and $c = z_1/z_2$, the equation becomes

$$\mathbb{E}\big[1_{\{Y(t_1) \le cY(t_2)\}} Y(t_2)\big] \exp\big(- \mathbb{E}\big[\big(Y(t_1) - cY(t_2)\big)^+\big]/z\big) = \exp\big(- 1/z\big).$$

This relation is satisfied for all $z, c > 0$ if and only if

$$\mathbb{E}\big[1_{\{Y(t_1) \le cY(t_2)\}} Y(t_2)\big] = 1 \quad \text{and} \quad \mathbb{E}\big[\big(Y(t_1) - cY(t_2)\big)^+\big] = 1, \quad c > 0.$$

Using $\mathbb{E}[Y(t_1)] = \mathbb{E}[Y(t_2)] = 1$, one can easily prove that this holds true if and only if

$$\mathbb{P}[Y(t_1) = 0 \text{ or } Y(t_2) = 0] = 1.$$

□

6.3 Proof of Theorem 6.1.2

6.3.1 A property of max-stable Markov processes

The following result is the central tool in our proof of Theorem 6.1.2. Note that no stationarity assumption is required at this stage.

Proposition 6.3.1. *Let $\eta = (\eta(t))_{t\in\mathbb{Z}}$ be a simple max-stable process with representation* (6.4)*. For $t, t' \in \mathbb{Z}$, we denote by $\alpha_{t,t'}$ the essential infimum of the random variable $Y(t')/Y(t)$ conditionally on $Y(t) > 0$, i.e.*

$$\alpha_{t,t'} = \inf\{c > 0;\ \mathbb{P}[Y(t')/Y(t) \leq c \mid Y(t) > 0] > 0\}. \tag{6.7}$$

If η satisfies the Markov property, then, for all $t_1 < t < t_2$,

$$\mathbb{P}[Y(t_1) = \alpha_{t,t_1} Y(t) \mid Y(t) > 0] = 1 \quad \textit{or} \quad \mathbb{P}[Y(t_2) = \alpha_{t,t_2} Y(t) \mid Y(t) > 0] = 1.$$

Proof of Proposition 6.3.1:
First note that the definition (6.7) entails

$$\mathbb{P}[Y(t') \geq \alpha_{t,t'} Y(t) \mid Y(t) > 0] = 1.$$

Hence, in order to prove Proposition 6.3.1, it is enough to prove

$$\mathbb{P}[Y(t_1) \leq \alpha_{t,t_1} Y(t) \mid Y(t) > 0] = 1 \quad \text{or} \quad \mathbb{P}[Y(t_2) \leq \alpha_{t,t_2} Y(t) \mid Y(t) > 0] = 1,$$

or equivalently that, for all $c_1 > \alpha_{t,t_1}$ and all $c_2 > \alpha_{t,t_2}$,

$$\mathbb{P}[Y(t_1) \leq c_1 Y(t) \mid Y(t) > 0] = 1 \ \text{ or } \ \mathbb{P}[Y(t_2) \leq c_2 Y(t) \mid Y(t) > 0] = 1. \tag{6.8}$$

We now prove Equation (6.8). We use the fact that the past and the future of a Markov chain are independent conditionally on the present. More formally, for all $t_1 < t < t_2$ and $z, z_1, z_2 > 0$,

$$\begin{aligned} & \mathbb{P}[\eta(t_1) \leq z_1, \eta(t_2) \leq z_2 \mid \eta(t) = z] \\ = \ & \mathbb{P}[\eta(t_1) \leq z_1 \mid \eta(t) = z]\, \mathbb{P}[\eta(t_2) \leq z_2 \mid \eta(t) = z]. \end{aligned}$$

Using the explicit expression for the conditional cumulative distribution function given in Proposition 6.2.7, this is equivalent to

$$\begin{aligned} & \mathbb{E}\big[1_{\{\frac{Y(t_1)}{z_1} \vee \frac{Y(t_2)}{z_2} \leq \frac{Y(t)}{z}\}} Y(t)\big] \exp\Big(- \mathbb{E}\Big[\Big(\frac{Y(t_1)}{z_1} \vee \frac{Y(t_2)}{z_2} - \frac{Y(t)}{z}\Big)^+\Big]\Big) \\ = \ & \mathbb{E}\big[1_{\{\frac{Y(t_1)}{z_1} \leq \frac{Y(t)}{z}\}} Y(t)\big] \exp\Big(- \mathbb{E}\Big[\Big(\frac{Y(t_1)}{z_1} - \frac{Y(t)}{z}\Big)^+\Big]\Big) \\ & \times \mathbb{E}\big[1_{\{\frac{Y(t_2)}{z_2} \leq \frac{Y(t)}{z}\}} Y(t)\big] \exp\Big(- \mathbb{E}\Big[\Big(\frac{Y(t_2)}{z_2} - \frac{Y(t)}{z}\Big)^+\Big]\Big). \end{aligned}$$

Using the formula $(a \vee b - c)^+ - (a-c)^+ - (b-c)^+ = (a \wedge b - c)^+$, this last equation simplifies into

$$\begin{aligned} & \mathbb{E}\big[1_{\{\frac{Y(t_1)}{z_1} \vee \frac{Y(t_2)}{z_2} \leq \frac{Y(t)}{z}\}} Y(t)\big] \exp\Big(\mathbb{E}\Big[\Big(\frac{Y(t_1)}{z_1} \wedge \frac{Y(t_2)}{z_2} - \frac{Y(t)}{z}\Big)^+\Big]\Big) \\ = \; & \mathbb{E}\big[1_{\{\frac{Y(t_1)}{z_1} \leq \frac{Y(t)}{z}\}} Y(t)\big] \mathbb{E}\big[1_{\{\frac{Y(t_2)}{z_2} \leq \frac{Y(t)}{z}\}} Y(t)\big]. \end{aligned}$$

Finally, setting $z_1 = c_1 z$ and $z_2 = c_2 z$ with $c_1, c_2, z > 0$, we obtain

$$\begin{aligned} & \mathbb{E}\big[1_{\{\frac{Y(t_1)}{c_1} \vee \frac{Y(t_2)}{c_2} \leq Y(t)\}} Y(t)\big] \exp\Big(\mathbb{E}\Big[\frac{1}{z}\Big(\frac{Y(t_1)}{c_1} \wedge \frac{Y(t_2)}{c_2} - Y(t)\Big)^+\Big]\Big) \\ = \; & \mathbb{E}\big[1_{\{\frac{Y(t_1)}{c_1} \leq Y(t)\}} Y(t)\big] \mathbb{E}\big[1_{\{\frac{Y(t_2)}{c_2} \leq Y(t)\}} Y(t)\big]. \end{aligned}$$

Note that the right hand side of this equality does not depend on $z > 0$ and is positive as soon as $c_1 > \alpha_{t,t_1}$ and $c_2 > \alpha_{t,t_2}$ (this is a simple consequence of the definition (6.7)). Then, the exponential factor in the left hand side must be constant and equal to 1. We deduce that, for all $c_1 > \alpha_{t,t_1}, c_2 > \alpha_{t,t_2}$,

$$\mathbb{E}\big[1_{\{\frac{Y(t_1)}{c_1} \vee \frac{Y(t_2)}{c_2} \leq Y(t)\}} Y(t)\big] = \mathbb{E}\big[1_{\{\frac{Y(t_1)}{c_1} \leq Y(t)\}} Y(t)\big] \mathbb{E}\big[1_{\{\frac{Y(t_2)}{c_2} \leq Y(t)\}} Y(t)\big] \tag{6.9}$$

and also

$$\mathbb{P}\Big[\frac{Y(t_1)}{c_1} \wedge \frac{Y(t_2)}{c_2} \leq Y(t)\Big] = 1. \tag{6.10}$$

Let us introduce the probability measure $\hat{\mathbb{P}}_t(\cdot) = \mathbb{E}[1_{\{\cdot\}} Y(t)]$ and the events

$$A_1 = \Big\{\frac{Y(t_1)}{c_1} \leq Y(t)\Big\} \quad \text{and} \quad A_2 = \Big\{\frac{Y(t_2)}{c_2} \leq Y(t)\Big\}.$$

With these notations, Equation (6.9) becomes

$$\hat{\mathbb{P}}_t[A_1 \cap A_2] = \hat{\mathbb{P}}_t[A_1]\hat{\mathbb{P}}_t[A_2]$$

and states that the events A_1 and A_2 are $\hat{\mathbb{P}}_t$-independent. On the other hand, Equation (6.10) yields $\mathbb{P}[A_1 \cup A_2] = 1$ which clearly implies $\hat{\mathbb{P}}_t[A_1 \cup A_2] = 1$. Taking the complementary set, we obtain $\hat{\mathbb{P}}_t[A_1^c \cap A_2^c] = 0$ and, from the independence of A_1 and A_2, $\hat{\mathbb{P}}_t[A_1^c]\hat{\mathbb{P}}_t[A_2^c] = 0$. Thus, we have $\hat{\mathbb{P}}_t[A_1^c] = 0$ or $\hat{\mathbb{P}}_t[A_1^c] = 0$. Finally, the probability measures $\hat{\mathbb{P}}_t[\,\cdot\,]$ and $\mathbb{P}[\,\cdot \mid Y(t) > 0]$ are equivalent in the sense that they have the same null sets. Hence, it holds

$$\mathbb{P}[A_1^c \mid Y(t) > 0] = 0 \quad \text{or} \quad \mathbb{P}[A_2^c \mid Y(t) > 0] = 0.$$

This is equivalent to Equation (6.8) and this concludes the proof of Proposition 6.3.1. $\square$

6.3.2 A characterization of max-AR(1) processes

We provide a simple characterization of max-autoregressive processes that will be useful for the proof of Theorem 6.1.2. We consider the cone of constant functions

$$D_1 = \{f \in \mathcal{F};\ \forall t \in \mathbb{Z},\ f(t) = f(0)\},$$

the cone of Dirac functions

$$D_0 = \{f \in \mathcal{F};\ \exists t_0 \in \mathbb{Z},\ \forall t \in \mathbb{Z},\ f(t) = f(t_0)1_{\{t=t_0\}}\},$$

and also, for $a \in (0,1)$, the cone

$$D_a = \{f \in \mathcal{F};\ \exists t_0 \in \mathbb{Z},\ \forall t \in \mathbb{Z},\ f(t) = f(t_0)a^{t-t_0}1_{\{t\geq t_0\}}\}.$$

Proposition 6.3.2. *Let η be a simple max-stable process with representation* (6.4) *and assume that η is stationary. Then, the following statements are equivalent :*
i) η has the same distribution as the max-AR(1) process X_a defined by (6.1)*,*
ii) $\mathbb{P}[Y \in D_a] = 1$.

Proof :
We denote by μ' the exponent measure of X_a. For $a \in [0,1)$, Equation (6.1) implies that $X_a = \vee_{n\in\mathbb{Z}} F_n f_a(\cdot - n)$ with $f_a(t) = (1-a)a^t 1_{\{t\geq 0\}}$, whence we deduce that μ' is given by

$$\mu'[A] = \sum_{n\in\mathbb{Z}} \int_0^\infty 1_{\{uf_a(\cdot-n)\in A\}} u^{-2} du, \quad A \subset \mathcal{F}_0 \text{ measurable.}$$

For $a = 1$, it holds $X_a = F_0 f_1$ with $f_1(t) \equiv 1$, so that

$$\mu'[A] = \int_0^\infty 1_{\{uf_1\in A\}} u^{-2} du, \quad A \subset \mathcal{F}_0 \text{ measurable.}$$

In both cases, μ' is clearly supported by the cone of functions D_a. This implies that if Y' is a spectral process associated to X_a, then $\mathbb{P}[Y' \in D_a] = 1$.

We now prove the implication $i) \Rightarrow ii)$. If η has the same distribution as the max-AR(1) process X_a, then the spectral processes Y and Y' are equivalent and Proposition 6.2.5 implies $\mathbb{P}[Y \in D_a] = 1$.

We finally prove the converse implication $ii) \Rightarrow i)$. We assume that $\mathbb{P}[Y \in D_a] = 1$ and we apply Lemma 6.2.6. Note that D_a is equal to the smallest shift invariant cone containing f_a and denoted by $C_{inv}(f_a)$. The spectral processes Y and Y' are Brown-Resnick stationary processes such that $\mathbb{P}[Y \in C_{inv}(f_a)] = \mathbb{P}[Y' \in C_{inv}(f_a)] = 1$. Lemma 6.2.6 entails that Y and Y' are equivalent, which means that η and X_a have the same distribution. $\square$

6.3.3 Proof of Theorem 6.1.2

Let η be a simple max-stable process with representation (6.4). We assume that η is stationary and satisfies the Markov property. According to Proposition 6.3.1 with $t = 0$, $t_1 = -1$ and $t_2 = 1$, it holds

$$\mathbb{P}[Y(-1) = \alpha_{0,-1}Y(0) \mid Y(0) > 0] = 1 \text{ or } \mathbb{P}[Y(1) = \alpha_{0,1}Y(0) \mid Y(0) > 0] = 1.$$

Two cases naturally appear :

- Case 1 : $\mathbb{P}[Y(1) = \alpha_{0,1}Y(0) \mid Y(0) > 0] = 1$.
  We will prove below that, in this case, η is a max-AR(1) process (6.1) with parameter $a = \alpha_{0,1}$. To this aim, we use the characterization of max-AR(1) processes given by Proposition 6.3.2 so that it is enough to prove $\mathbb{P}[Y \in D_a] = 1$. Note that $a \in [0,1]$ since $a = \alpha_{0,1} = \mathbb{E}[Y(1)1_{\{Y(0)>0\}}]$.
- Case 2 : $\mathbb{P}[Y(-1) = \alpha_{0,-1}Y(0) \mid Y(0) > 0] = 1$.
  We prove that, in this case, η is a time reversed max-AR(1) process (6.2) with parameter $a = \alpha_{0,-1}$. This is easily deduced from case 1 since the time reversed process $\check{\eta} = (\eta(-t))_{t\in\mathbb{Z}}$ is a stationary simple max-stable process satisfying the Markov property. The associated spectral process $\check{Y} = (Y(-t))_{t\in\mathbb{Z}}$ satisfies $\mathbb{P}[\check{Y}(1) = \alpha_{0,-1}\check{Y}(0) \mid \check{Y}(0) > 0] = 1$, so that $\check{\eta}$ is a max-AR(1) process with parameter $a = \alpha_{0,-1}$.

Thanks to the discussion above, the proof of Theorem 6.1.2 is reduced to the proof of the following statement :

$$\text{If } \mathbb{P}[Y(1) = aY(0) \mid Y(0) > 0] = 1, \text{ then } \mathbb{P}[Y \in D_a] = 1. \tag{6.11}$$

We consider three different cases : $a \in (0,1)$, $a = 0$ and $a = 1$.

Proof of (6.11) **in the case** $a \in (0,1)$ **:**
We define the cone $C \subset \mathcal{F}$ by

$$C = \{f \in \mathcal{F};\ f(0) > 0 \Rightarrow f(1) = af(0)\}.$$

The property $\mathbb{P}[Y(1) = aY(0) \mid Y(0) > 0] = 1$ implies $\mathbb{P}[Y \in C] = 1$. Since η is stationary, Y is Brown-Resnick stationary and Proposition 6.2.5 implies

$$\mathbb{P}\Big[Y \in \bigcap_{s\in\mathbb{Z}} \theta_s(C)\Big] = 1. \tag{6.12}$$

Clearly, $\bigcap_{s\in\mathbb{Z}} \theta_s(C)$ is equal to the set of functions

$$\big\{f \in \mathcal{F};\ \forall s \in \mathbb{Z}, f(s) > 0 \Rightarrow f(s+1) = af(s)\big\}.$$

For such a function f, we easily prove by induction that $f(t_0) > 0$ implies $f(t) = f(t_0)a^{t-t_0}$ for all $t > t_0$. Then, if $t_0 = \min\{t \in \mathbb{Z}; f(t) > 0\} > -\infty$, $f(t) = f(t_0)a^{t-t_0}1_{\{t\geq t_0\}}$ for all $t \in \mathbb{Z}$, and $f \in D_a$. Otherwise, if $t_0 = \min\{t \in \mathbb{Z}; f(t) > 0\} = -\infty$, $f(t) = f(0)a^t$ for all $t \in \mathbb{Z}$ and f belongs to the cone D'_a generated by the power function $t \mapsto a^t$. This proves

$$\bigcap_{s\in\mathbb{Z}} \theta_s(C) = D_a \cup D'_a.$$

So Equation (6.12) is equivalent to $\mathbb{P}[Y \in D_a \cup D'_a] = 1$. In order to prove $\mathbb{P}[Y \in D_a] = 1$, it remains to prove that $\mathbb{P}[Y \in D'_a \setminus D_a] = 0$. Note that all function $f \in D'_a \setminus D_a$ is of the form $f(t) = ua^t$, $u > 0$ and satisfies $\lim_{t\to-\infty} f(t) = +\infty$. Hence,

$$\mathbb{P}[Y \in D'_a \setminus D_a] \leq \mathbb{P}[\lim_{t\to-\infty} Y(t) = +\infty] = 0$$

where the last equality is a consequence of (6.3). This proves Equation (6.11).

Proof of (6.11) **in the case** $a = 1$ **:**
First we prove that

$$\mathbb{P}[Y(1) = Y(0) \mid Y(0) > 0] = 1 \quad \text{implies} \quad \mathbb{P}[Y(-1) = Y(0) \mid Y(0) > 0] = 1.$$

To see this, we note that $\mathbb{P}[Y(1) = Y(0) \mid Y(0) > 0] = 1$ if and only if $\mathbb{P}[Y \in C] = 1$ with $C = \{f \in \mathcal{F}; f(0) > 0 \Rightarrow f(1) = f(0)\}$. Since Y is Brown-Resnick stationary, Proposition 6.2.5 implies $\mathbb{P}[Y(\cdot - 1) \in C] = 1$, which yields $\mathbb{P}[Y(0) = Y(-1) \mid Y(-1) > 0] = 1$. Then, Equation (6.3) entails

$$\mathbb{E}\big[Y(-1)1_{\{Y(-1)=Y(0)\}}\big] = \mathbb{E}\big[Y(0)1_{\{Y(-1)=Y(0)\}}\big] = 1.$$

We deduce $\mathbb{E}\big[Y(0)1_{\{Y(-1)\neq Y(0)\}}\big] = 0$ which implies

$$\mathbb{P}[Y(-1) = Y(0) \mid Y(0) > 0] = 1.$$

Consider the cone

$$C = \{f \in \mathcal{F};\ f(0) > 0 \Rightarrow f(1) = f(-1) = f(0)\}.$$

The conditions

$$\mathbb{P}[Y(1) = Y(0) \mid Y(0) > 0] = \mathbb{P}[Y(-1) = Y(0) \mid Y(0) > 0] = 1$$

implies $\mathbb{P}[Y \in C] = 1$, whence Proposition 6.2.5 yields

$$\mathbb{P}\Big[Y \in \bigcap_{s\in\mathbb{Z}} \theta_s(C)\Big] = 1.$$

Clearly, $\bigcap_{s\in\mathbb{Z}} \theta_s(C)$ is equal to the cone of functions

$$\big\{f \in \mathcal{F};\ \forall s \in \mathbb{Z}, f(s) > 0 \Rightarrow f(s+1) = f(s-1) = f(s)\big\}.$$

One can easily prove by induction that this is the cone D_1 of constant functions. This proves Equation (6.11).

Proof of (6.11) **in the case** $a = 0$ **:**
According to Proposition 6.2.8, $\mathbb{P}[Y(1) = 0 \mid Y(0) > 0] = 1$ if and only if $\eta(0)$ and $\eta(1)$ are independent. Let $t \geq 2$. By the Markov property, $\eta(0)$ and $\eta(t)$ are independent conditionally on $\eta(1)$. But since $\eta(0)$ and $\eta(1)$ are independent, this implies that $\eta(0)$ and $\eta(t)$ are independent. Hence $\eta(0)$ and $\eta(t)$ are independent for all $t \geq 1$ and by the stationarity of η, $\eta(t)$ and $\eta(t')$ are independent for all $t \neq t'$. Using Proposition 6.2.8 again, we deduce

$$\mathbb{P}[Y(t') = 0 \mid Y(t) > 0] = 1 \quad \text{for all } t \neq t',$$

and also

$$\mathbb{P}\Big[\forall t' \neq t,\ Y(t') = 0 \mid Y(t) > 0\Big] = 1.$$

This implies that the set where Y is non zero has almost surely at most one point. Equivalently, $\mathbb{P}[Y \in D_0] = 1$ and Equation (6.11) is proved.

6.4 Continuous time setting

We consider in this section an extension of Theorem 6.1.2 to the continuous time framework. For $a \in (0,1)$, we denote by $g_a(t) = -\log(a)a^t 1_{\{t \geq 0\}}$ the power function. The constant $-\log(a)$ ensures the normalization $\int_{\mathbb{R}} g_a(t)dt = 1$. We consider the moving maximum process

$$Z_a(t) = \bigvee_{i \geq 1} U_i g_a(t - T_i), \quad t \in \mathbb{R}, \tag{6.13}$$

where $\{(U_i, T_i);\ i \geq 1\}$ is a Poisson point process on $(0, +\infty) \times \mathbb{R}$ with intensity $u^{-2}dudt$. The time reversed process $\check{Z}_a$ is defined similarly by

$$\check{Z}_a(t) = \bigvee_{i \geq 1} U_i \check{g}_a(t - T_i), \quad t \in \mathbb{R}, \tag{6.14}$$

with $\check{g}_a(t) = -\log(a)a^{-t}1_{\{t<0\}} = g_a(-t^-)$. We use here a slightly different notion of time reversal so that the function $\check{g}_a$ is càd-làg.
For $a = 1$, we define $Z_1 = \check{Z}_1$ a process with constant path and such $Z_1(0)$ has a standard 1-Fréchet distribution.

Lemma 6.4.1. *The processes Z_a and $\check{Z}_a$ are stationary simple max-stable processes satisfying the Markov property and with càd-làg sample paths.*

Proof Proof of Lemma 6.4.1:
The result is straightforward when $a = 1$. For $a \in (0,1)$, the process Z_a is a moving maximum process with shape function g_a satisfying $\int_{\mathbb{R}} g_a(t)dt = 1$ and is hence a stationary simple max-stable process. The sample paths are càd-làg because the shape function g_a is càd-làg and satisfies

$\int_{\mathbb{R}} \sup_{|z|\le M} g_a(t+z)dt < \infty$ for all $M > 0$.
Straightforward computations yield that for any $t \in \mathbb{R}$

$$Z_a(t+s) = a^s Z_a(t) \bigvee F_a(t,s), \quad s \ge 0, \tag{6.15}$$

with

$$F_a(t,s) = \bigvee_{i\ge 1} U_i g_a(t+s-T_i) 1_{\{T_i > t\}}. \tag{6.16}$$

Note that for $t' \le t$, $Z_a(t')$ depends only on the points (U_i, T_i) such that $T_i \le t$ while $F_a(t,s)$ depends only on the points (U_i, T_i) such that $t + s \ge T_i > t$. This implies that $(Z_a(t'))_{t'\le t}$ and $(F_a(t,s))_{s\ge 0}$ are independent processes. This together with Equation (6.15) implies that the process Z_a satisfies the Markov property.
The similar statements for the time reversed process $\check{Z}_a$ are proved in the same way and we omit the details. □

Theorem 6.1.2 extends to continuous time processes as follows.

Theorem 6.4.2. *Any stationary simple max-stable process $\eta = (\eta(t))_{t\in\mathbb{R}}$ with cád-làg sample paths and satisfying the Markov property is equal in distribution to Z_a or $\check{Z}_a$ for some $a \in (0,1]$.*

Equality in distribution is meant in the sense of equality of laws in the Skohorod space $\mathcal{D}(\mathbb{R},\mathbb{R})$ with the J_1-topology. If the max-stable Markov process is η is not supposed càd-làg but only continuous in probability, the result still holds in the sense of equality of the finite dimensional distributions.

For the proof of Theorem 6.4.2, the following Lemma will be useful.

Lemma 6.4.3. *For all $\varepsilon > 0$, the discrete time process $Z_a^\varepsilon = (Z_a(\varepsilon t))_{t\in\mathbb{Z}}$ is a max-AR(1) process with parameter a^ε.*

Proof Proof of Lemma 6.4.3:
The random variable $F_a(t,s)$ given by (6.16) has a 1-Fréchet distribution with scale parameter $1 - a^s$ since

$$\begin{aligned} \mathbb{P}[F_a(t,s) \le x] &= \mathbb{P}\Big[\bigvee_{i\ge 1} U_i g_a(t+s-T_i) 1_{\{T_i>t\}} \le x\Big] \\ &= \exp\Big(-\int_{\mathbb{R}}\int_0^{+\infty} 1_{\{u(-\log(a))a^{t+s-\tau}>x\}} 1_{\{t<\tau\le t+s\}} u^{-2} du d\tau\Big) \\ &= \exp\big(-(1-a^s)/x\big). \end{aligned}$$

Using this, one proves easily that the random variables $F_t = F_a(\varepsilon t, \varepsilon)/(1-a^\varepsilon)$, $t \in \mathbb{Z}$, are i.i.d. with standard Fréchet distribution. Equation (6.15) entails

$$Z_a^\varepsilon(t+1) = a^\varepsilon Z_a^\varepsilon(t) \bigvee (1-a^\varepsilon) F_t, \quad t \in \mathbb{Z}$$

so that Z_a^ε is a max-AR(1) process with parameter a^ε. □

Proof Proof of Theorem 6.4.2:
The discrete time process $\eta^1 = (\eta(t))_{t\in\mathbb{Z}}$ extracted from the continuous time process η is a stationary simple max-stable process on $\mathbb{Z}$ satisfying the Markov property. By Theorem 6.1.2, it is equal in distribution either to a max-AR(1) process X_a with $a \in [0,1]$ or a time reversed max-AR(1) process $\check{X}_a$ with $a \in (0,1)$.

– In the case $\eta^1 \stackrel{d}{=} X_a$ with $a \in (0,1]$, we prove that $\eta \stackrel{d}{=} Z_a$.
  The process $\eta^{1/n} = (\eta(t/n))_{t\in\mathbb{Z}}$ is stationary simple max-stable and Markov. Theorem 6.1.2 entails that $\eta^{1/n}$ is either a max-AR(1) process X_{a_n} with $a_n \in [0,1]$ or a time reversed max-AR(1) process $\check{X}_{a_n}$ with $a_n \in (0,1)$. Using the relation $(\eta^1(t))_{t\in\mathbb{Z}} = (\eta^{1/n}(nt))_{t\in\mathbb{Z}}$, we prove easily that $\eta^{1/n}$ must be a max-AR(1) process with parameter $a_n = a^{1/n}$. Indeed, in all other cases, the process $(\eta^{1/n}(nt))_{t\in\mathbb{Z}}$ is not a max-AR(1) process with parameter a. By Lemma 6.4.3, the process $Z_a^{1/n}$ is also a max-AR(1) process with parameter $a^{1/n}$ so that the processes $(\eta(t/n))_{t\in\mathbb{Z}}$ and $(Z_a(t/n))_{t\in\mathbb{Z}}$ have the same distribution. Since this holds true for all $n \geq 1$, we easily see that, for all rational numbers $t_1,\ldots,t_p \in \mathbb{Q}$, the random vectors $(\eta(t_1),\ldots,\eta(t_p))$ and $(Z_a(t_1),\ldots,Z_a(t_p))$ have the same distribution. Together with the property that both η and Z_a have càd-làg sample paths, this implies that η and Z_a have the same distribution in the Skohorod space $\mathcal{D}(\mathbb{R},\mathbb{R})$(see Billingsley [8] theorem 14.5).
– We show that the case $\eta^1 \stackrel{d}{=} X_0$ can not occur.
  Indeed, if $\eta^1 \stackrel{d}{=} X_0$, it holds also $\eta^{1/n} \stackrel{d}{=} X_0$ for all $n \geq 1$. This proves that the random variables $\eta(t), t \in \mathbb{Q}$ are independent with standard Fréchet distribution. This contradicts the fact that η has cád-làg sample paths since the difference $\eta(1/n)-\eta(0)$ should converge in law to zero as $n \to +\infty$.
– In the case $\eta^1 \stackrel{d}{=} \check{X}_a$ with $a \in (0,1)$, we prove that $\eta \stackrel{d}{=} \check{Z}_a$.
  Indeed, the time reversed process $\check{\eta} = (\eta(-t^-))_{t\in\mathbb{Z}}$ is then stationary simple max-stable and Markov and such that $(\check{\eta}(t))_{t\in\mathbb{Z}}$ is a max-AR(1) process with parameter a. Hence $\check{\eta}$ and Z_a have the same distribution, whence η and $\check{Z}_a$ have the same distribution.

□

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